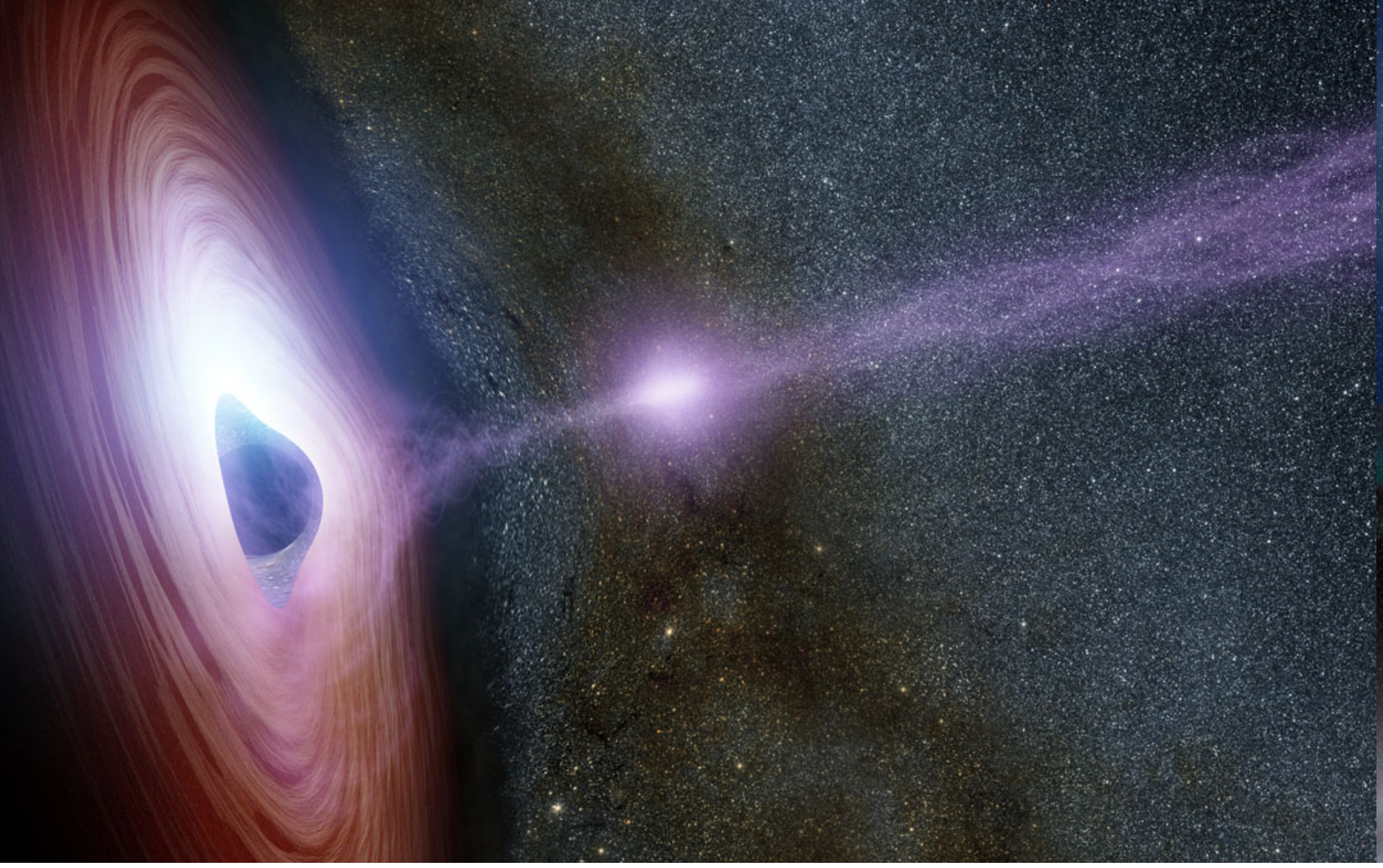

Black Holes

The Theory

Dr Dan Wilkins
Saint Mary's University, Halifax, NS

CRAQ Compact Objects Summer School, Montréal, August 2016

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Overview

- Part 1: Developing General Relativity
- Part 2: Black holes and their properties
- Part 3: Accretion, from simple theory to messy reality
- Part 4: Using the theory to get close to black holes

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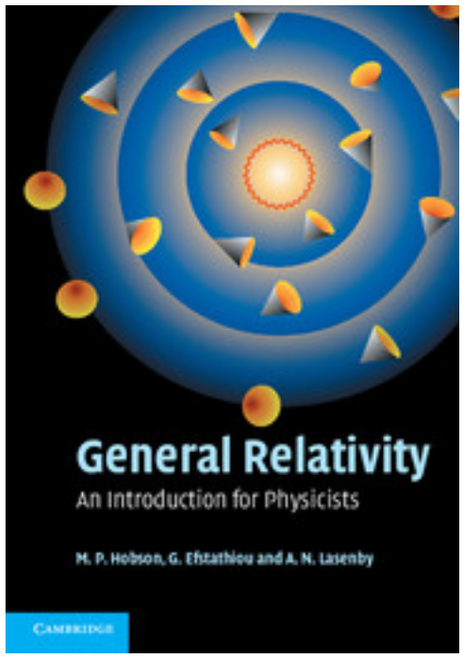
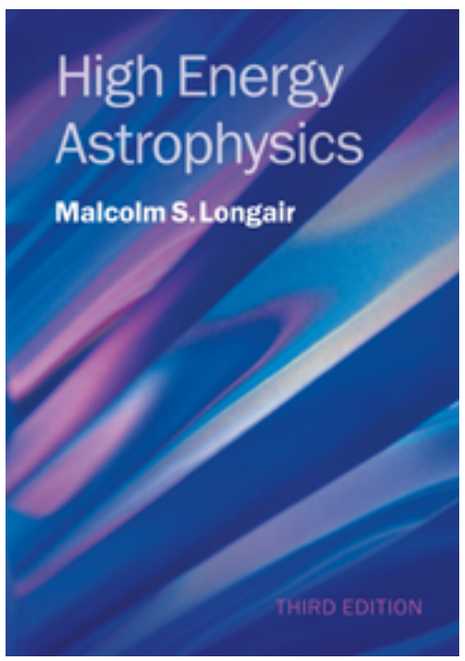
Recommended Reading



General Relativity, An Introduction for Physicists
M.P. Hobson, G. Efstathiou and A.N. Lasenby (Cambridge, 2006)



High Energy Astrophysics
M.S. Longair (3rd Ed, Cambridge, 2011)

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This lecture is interactive!

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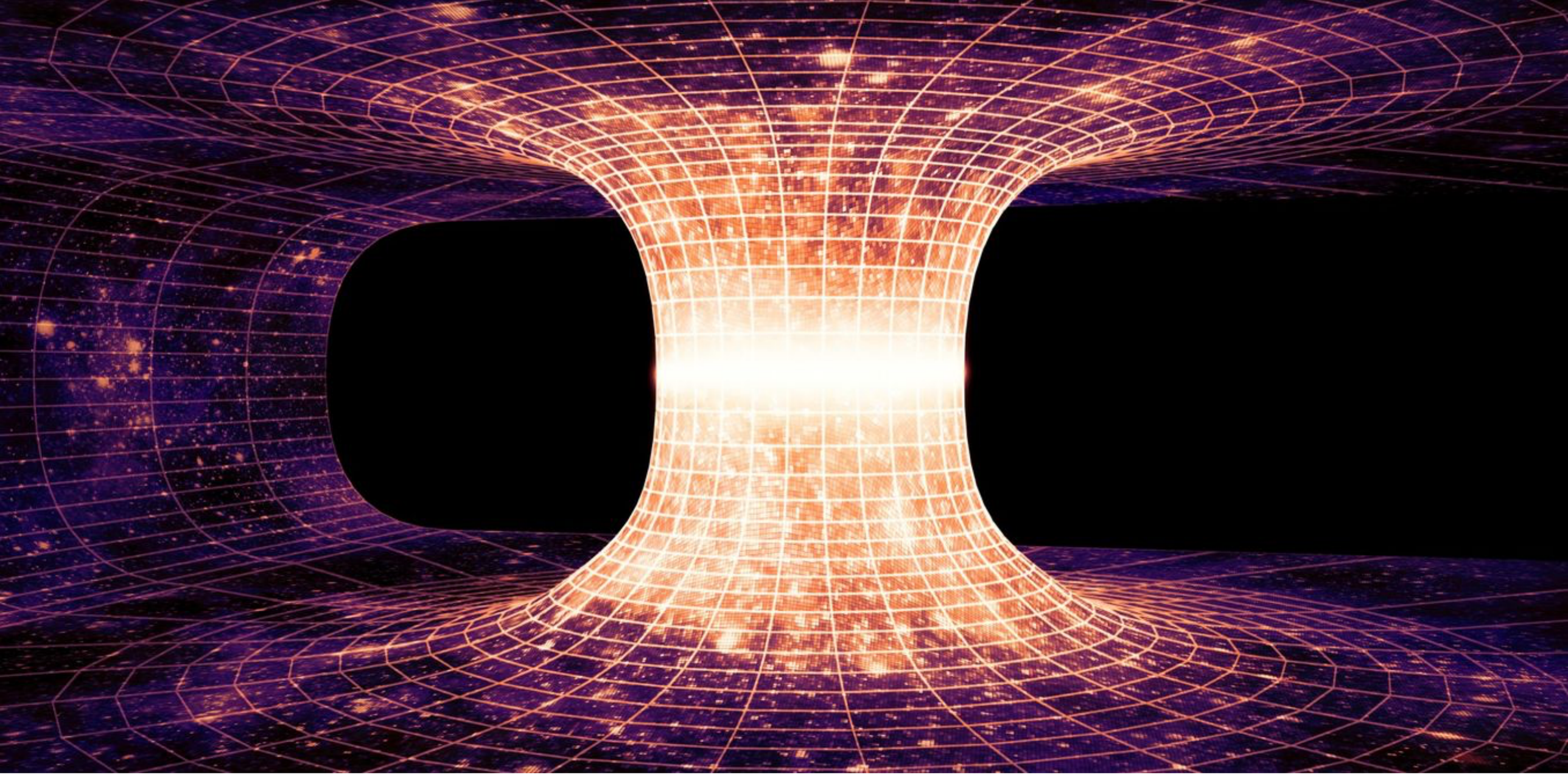

Developing General Relativity

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General Relativity – Why Bother?

- Newtonian gravity

$$\vec{f} = m\vec{g} = -m\vec{\nabla}\Phi$$

- Poisson's equation

$$\vec{\nabla}^2\Phi = 4\pi G\rho$$

- No explicit time dependence — potential responds instantaneously to change in matter density over all space

  - Information travels faster than the speed of light!

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The Equivalence Principle

- Gravitational mass (coupling to gravitational field) vs. inertial mass (resistance to applied force)

$$\vec{f} = -m_{\mathrm{G}} \vec{\nabla} \Phi \qquad\qquad \frac{d^2 \vec{x}}{dt^2} = -\frac{m_{\mathrm{G}}}{m_{\mathrm{I}}} \vec{\nabla} \Phi$$

- Well-established experimental fact that ratio $m_{\mathrm{G}}/m_{\mathrm{I}}$ is constant for *all particles*

- Trajectory of a particle through a gravitational field *is the same, regardless of the nature of the particle*

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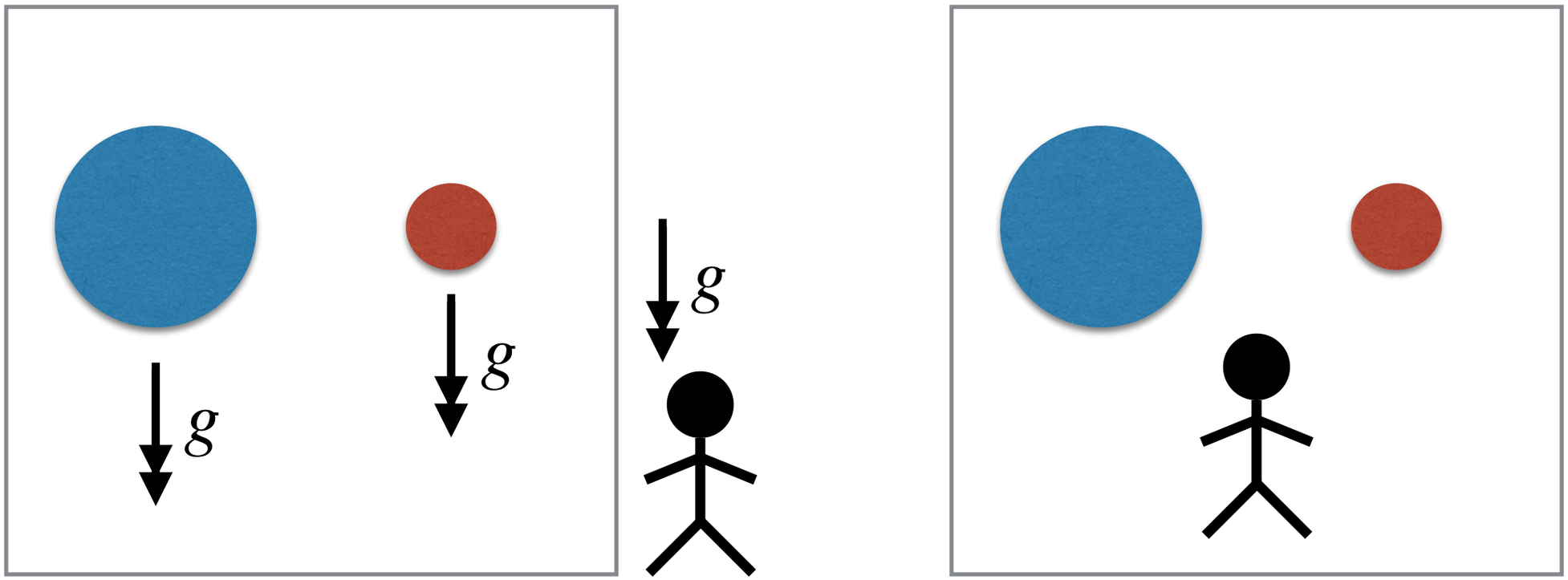
Einstein's 'Elevator'



In a freely-falling (non-rotating) laboratory occupying a small region of spacetime, the laws of physics are those of special relativity — the (strong) equivalence principle

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Special Relativity

- No preferred frame of reference
- Time passes differently for different observers

$$\Delta t' = \gamma \Delta t \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

- Length contraction

$$\Delta x' = \frac{\Delta x}{\gamma}$$

- Mass/energy equivalence $E = \gamma m_0 c^2$
- Information cannot propagate faster than c

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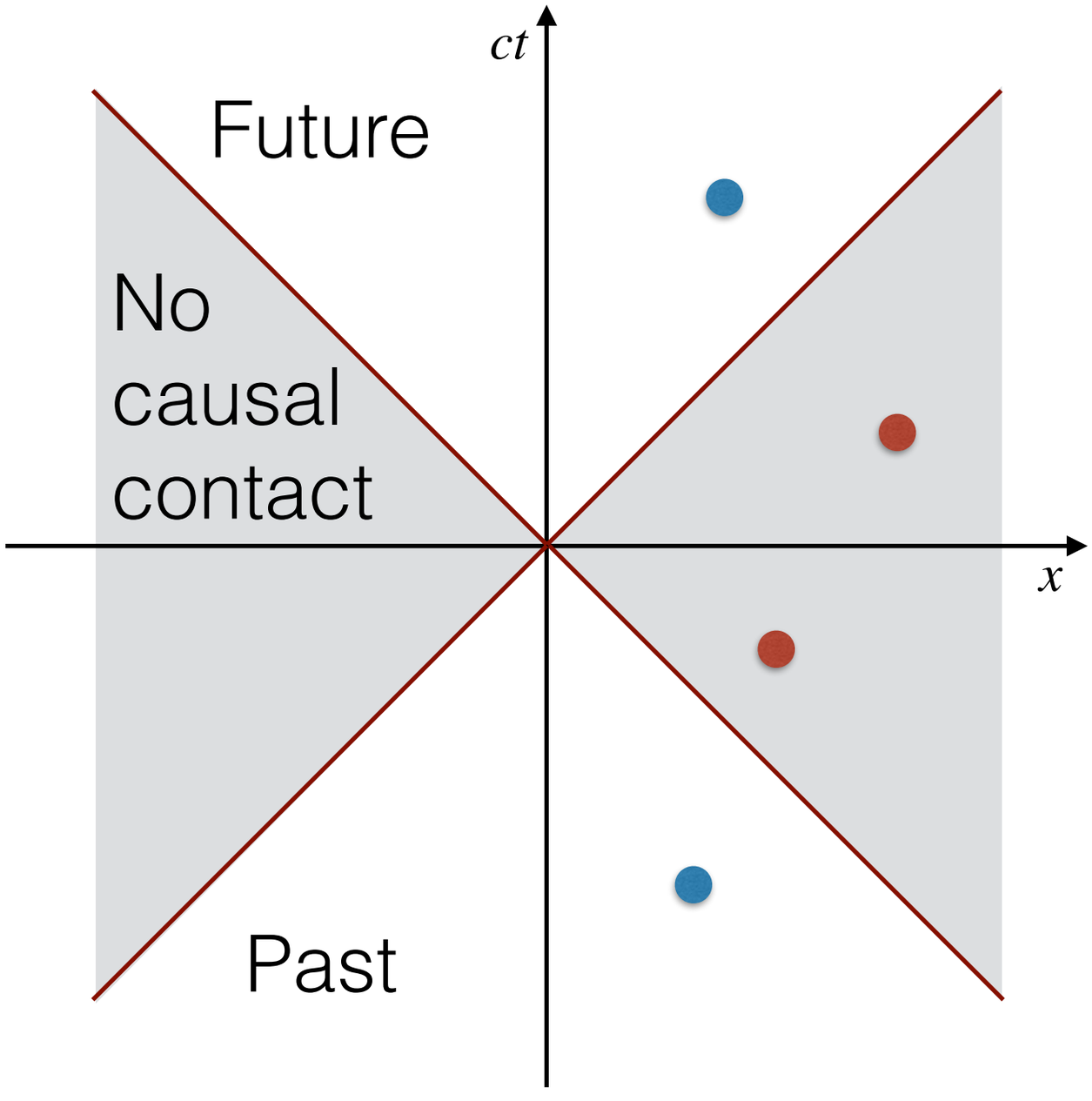
Spacetime Diagrams



- Timelike

$$c^2\Delta t^2 - \Delta x^2 > 0$$

- Spacelike

$$c^2\Delta t^2 - \Delta x^2 < 0$$

- Null (photon)

$$c^2\Delta t^2 - \Delta x^2 = 0$$

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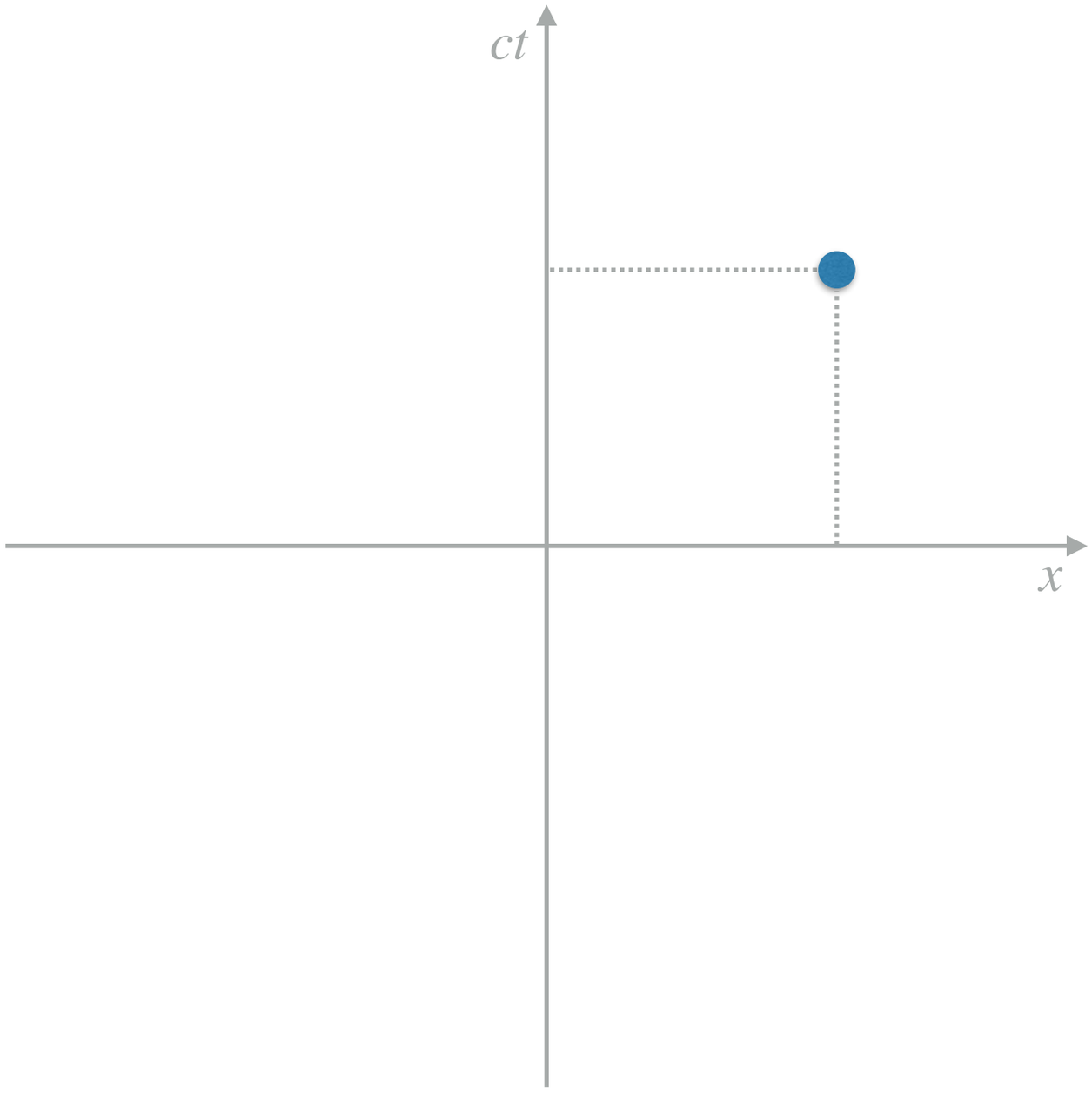
Lorentz Transformations



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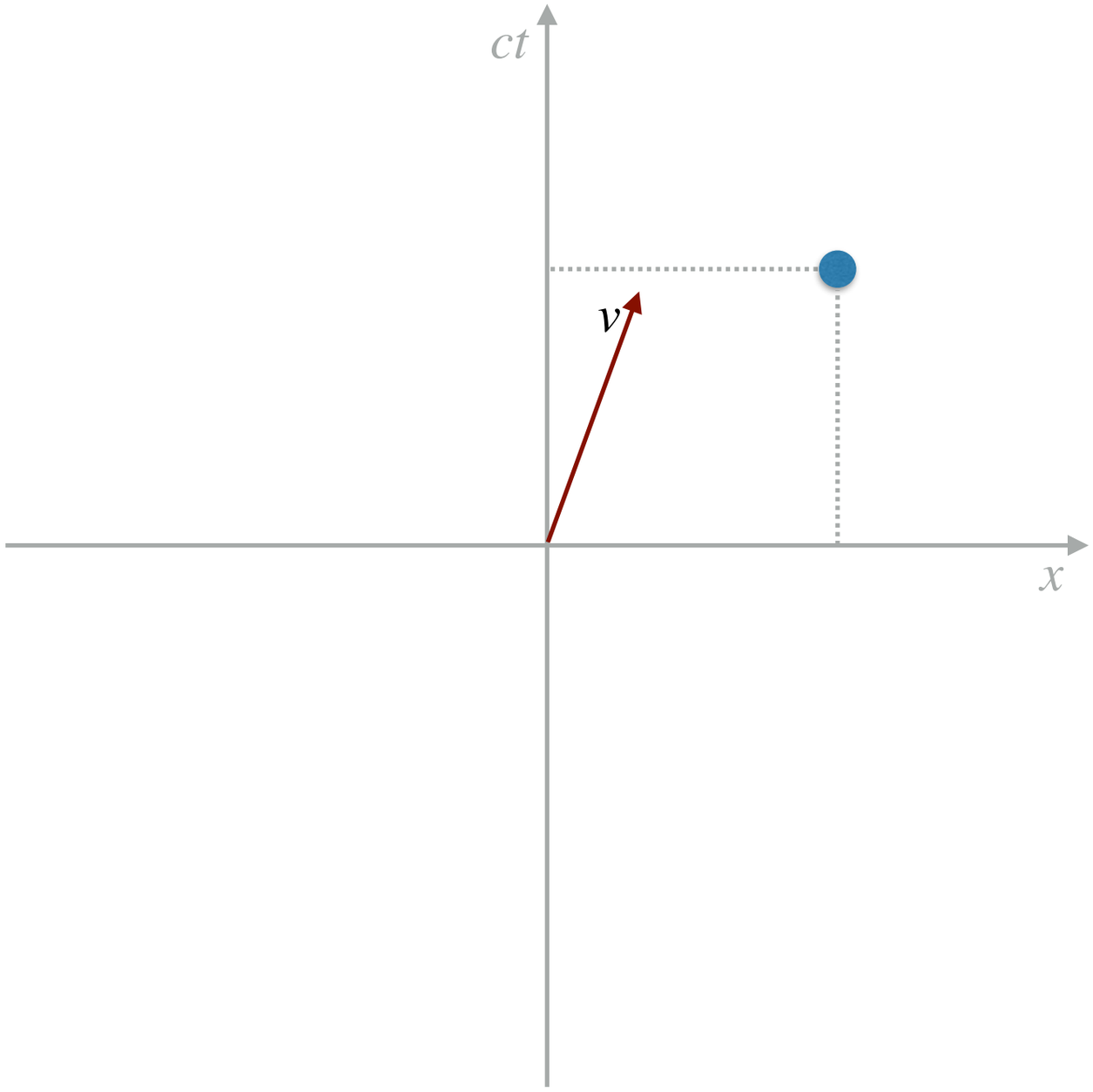
Lorentz Transformations



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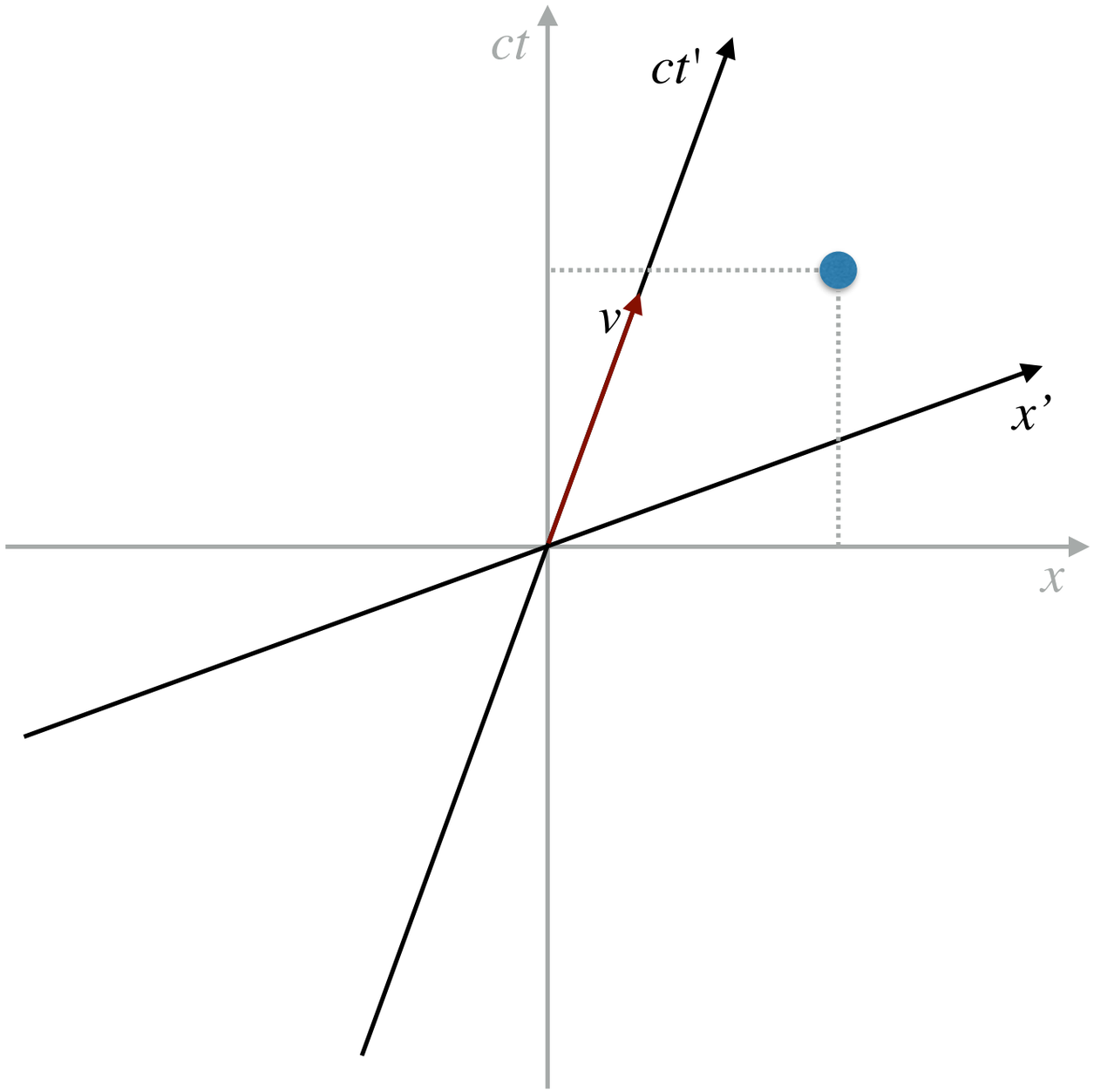

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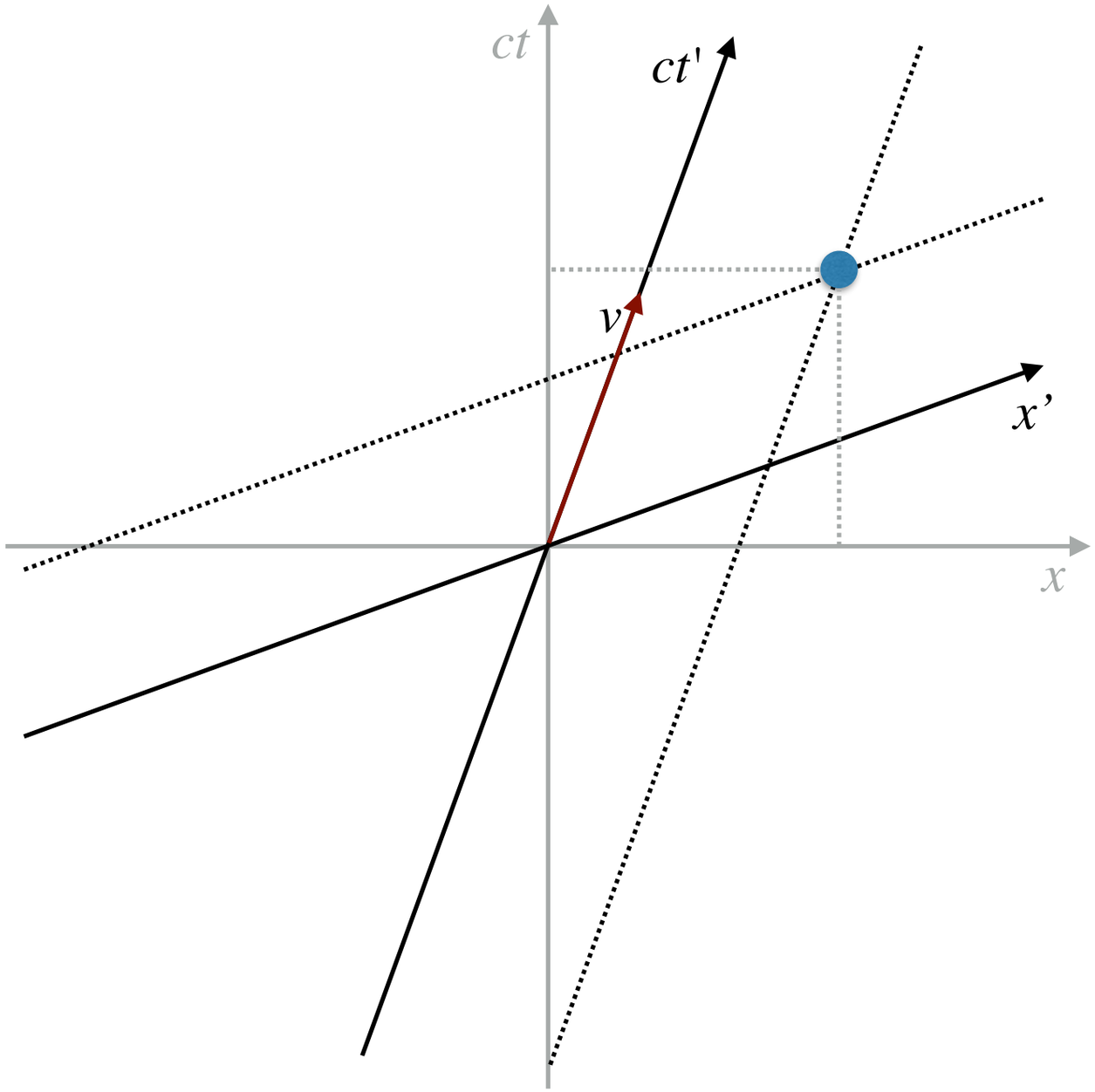

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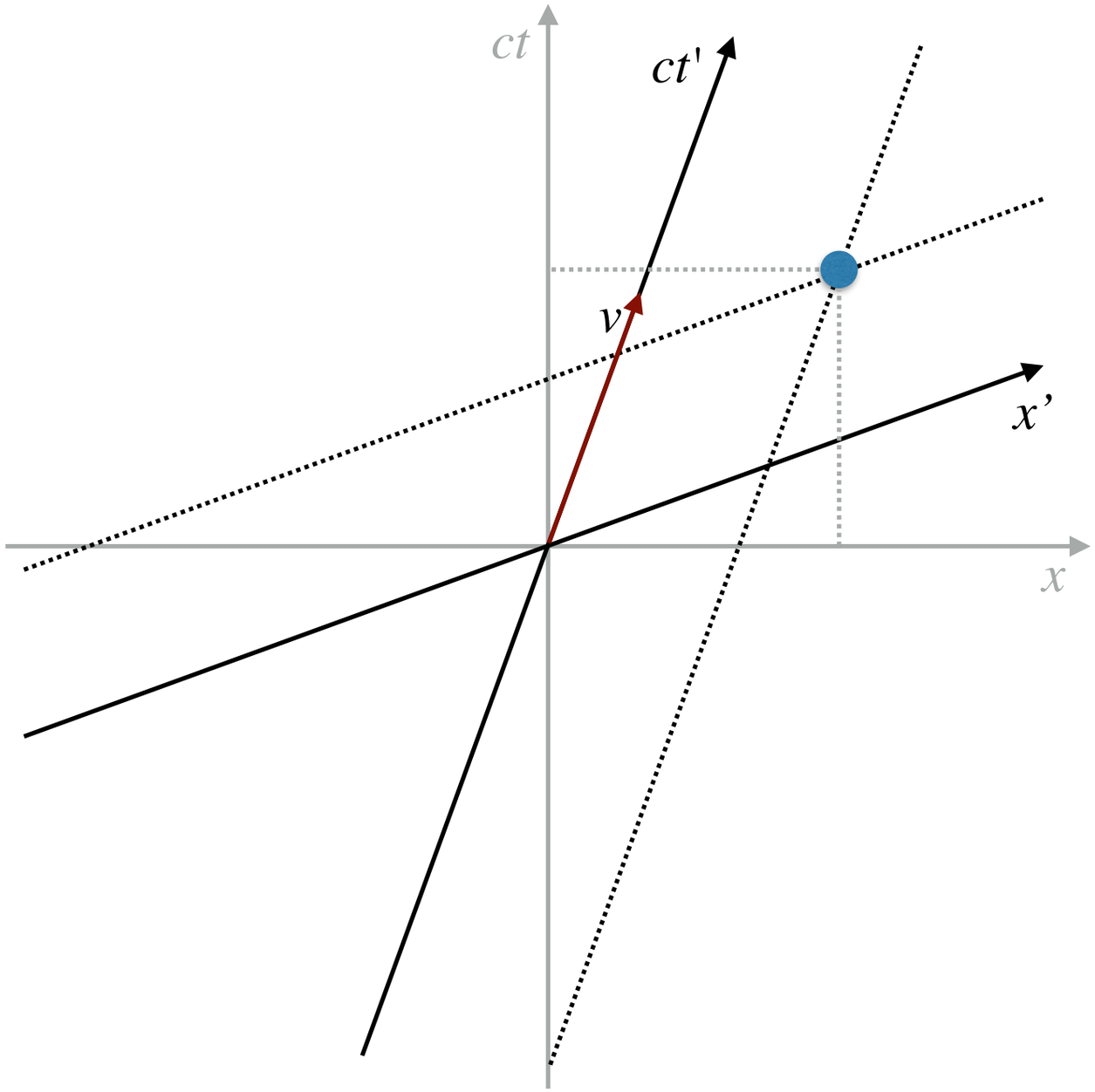
Lorentz Transformations



$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

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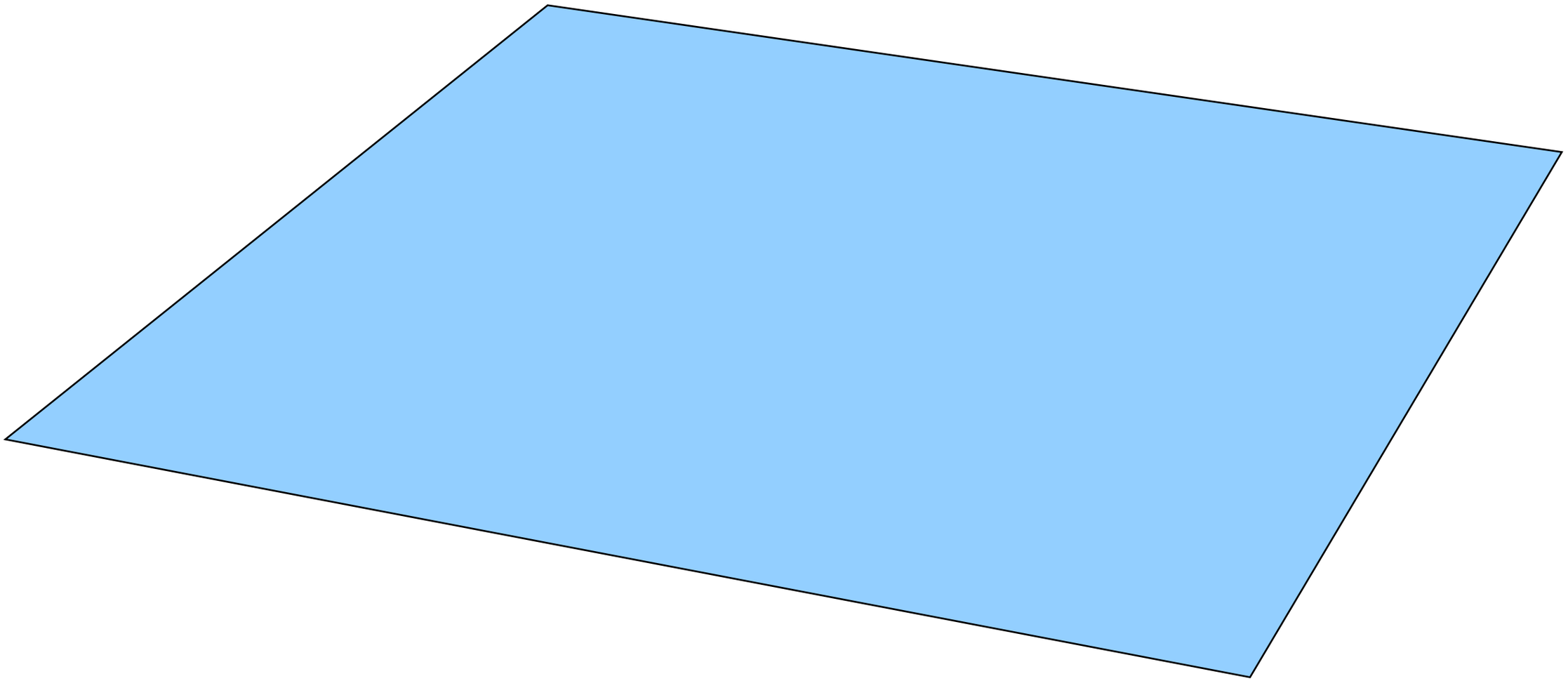
Gravity as Spacetime Curvature

“Matter tells space how to curve. Space tells matter how to move” — John Archibald Wheeler



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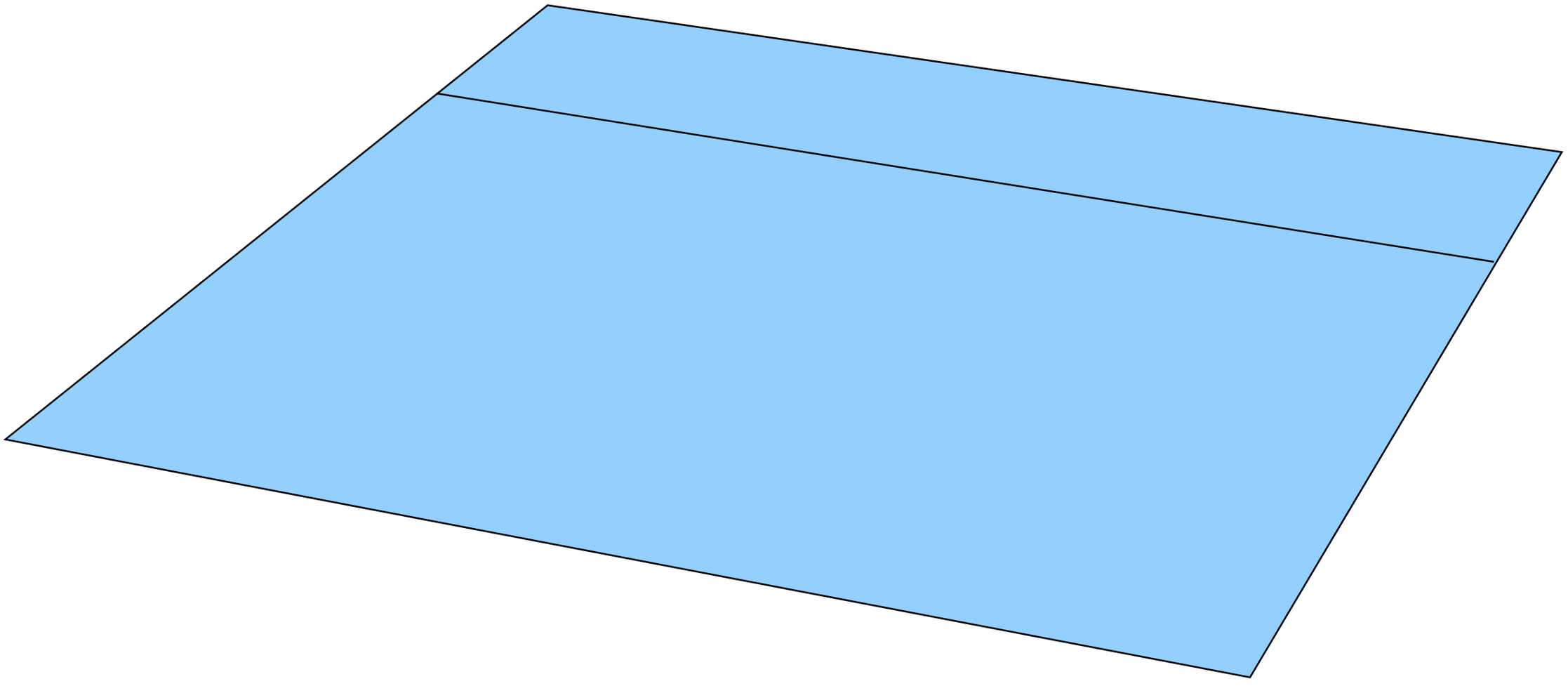

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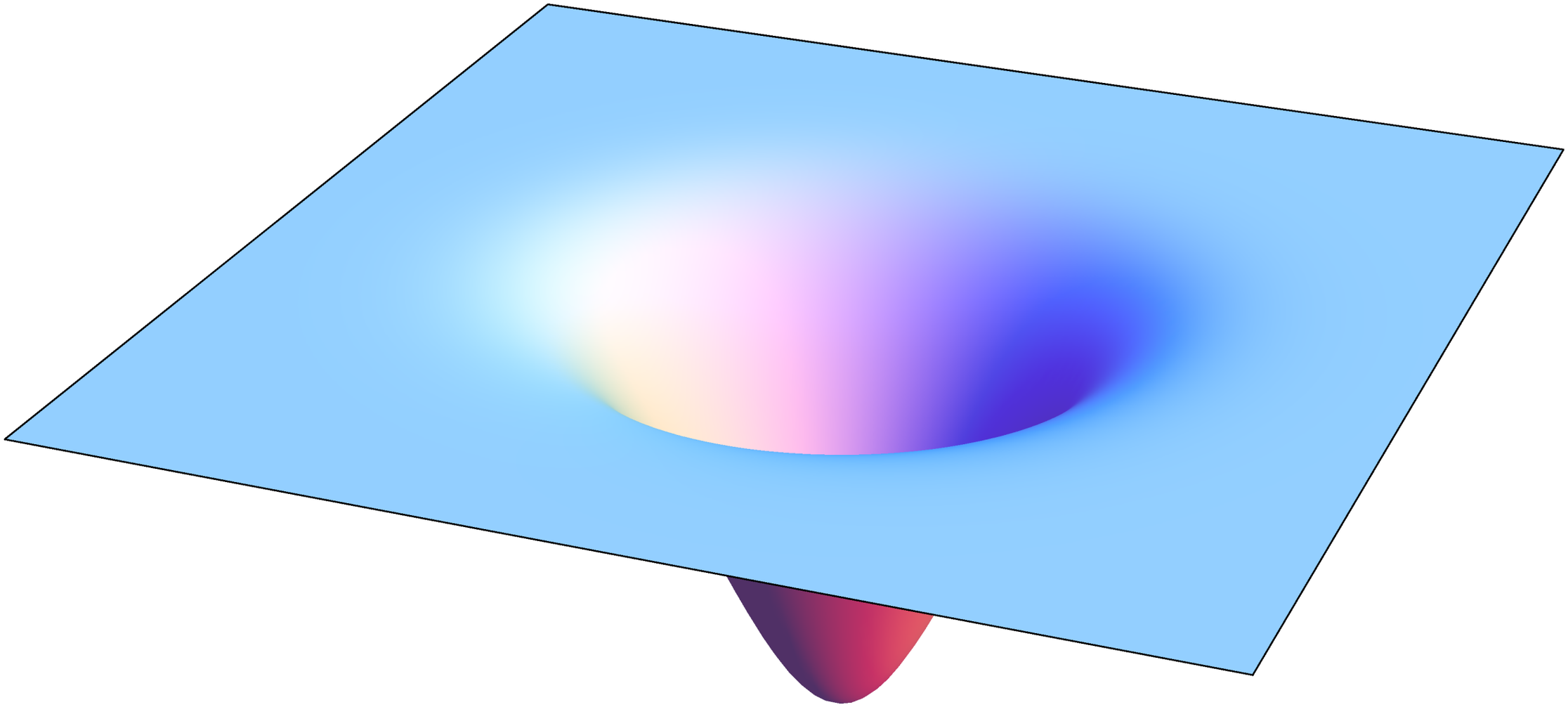

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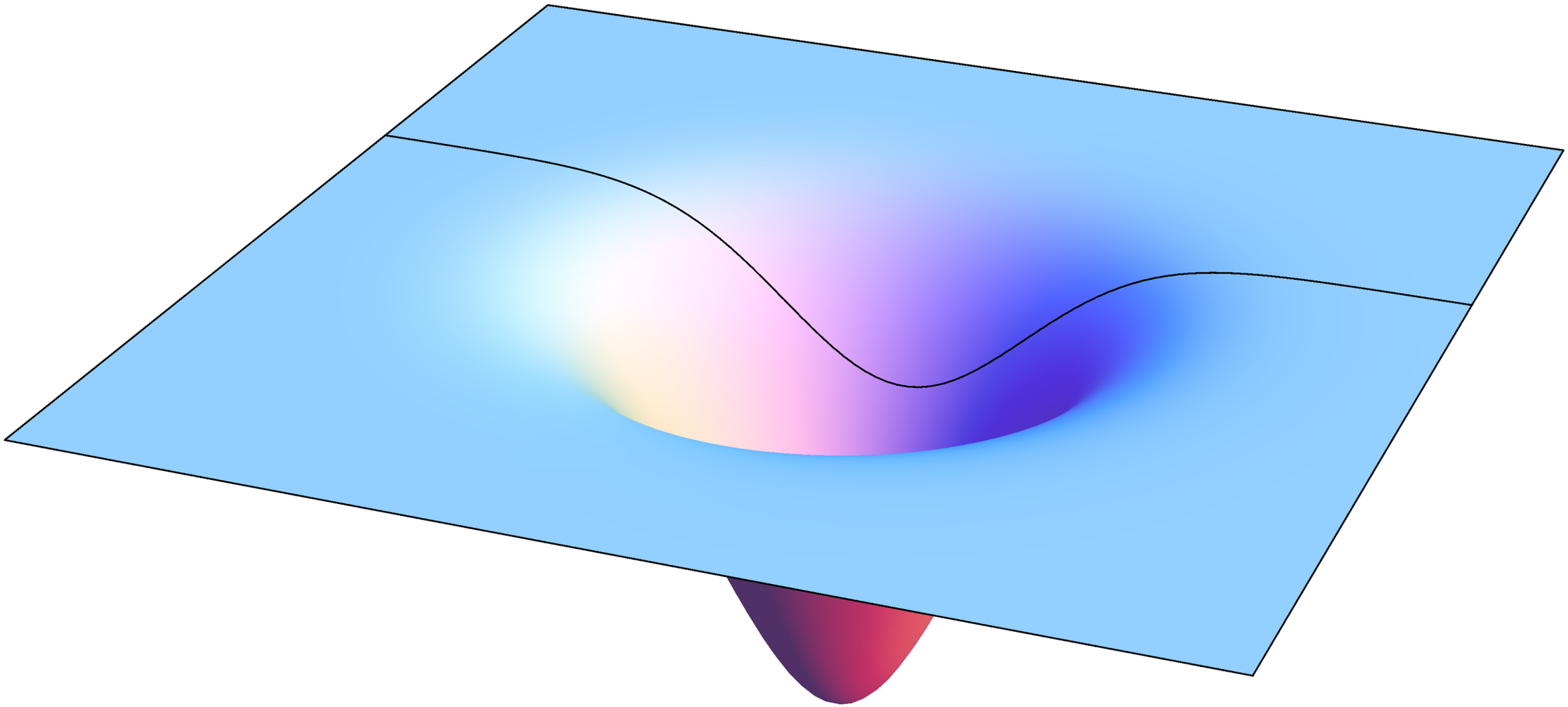

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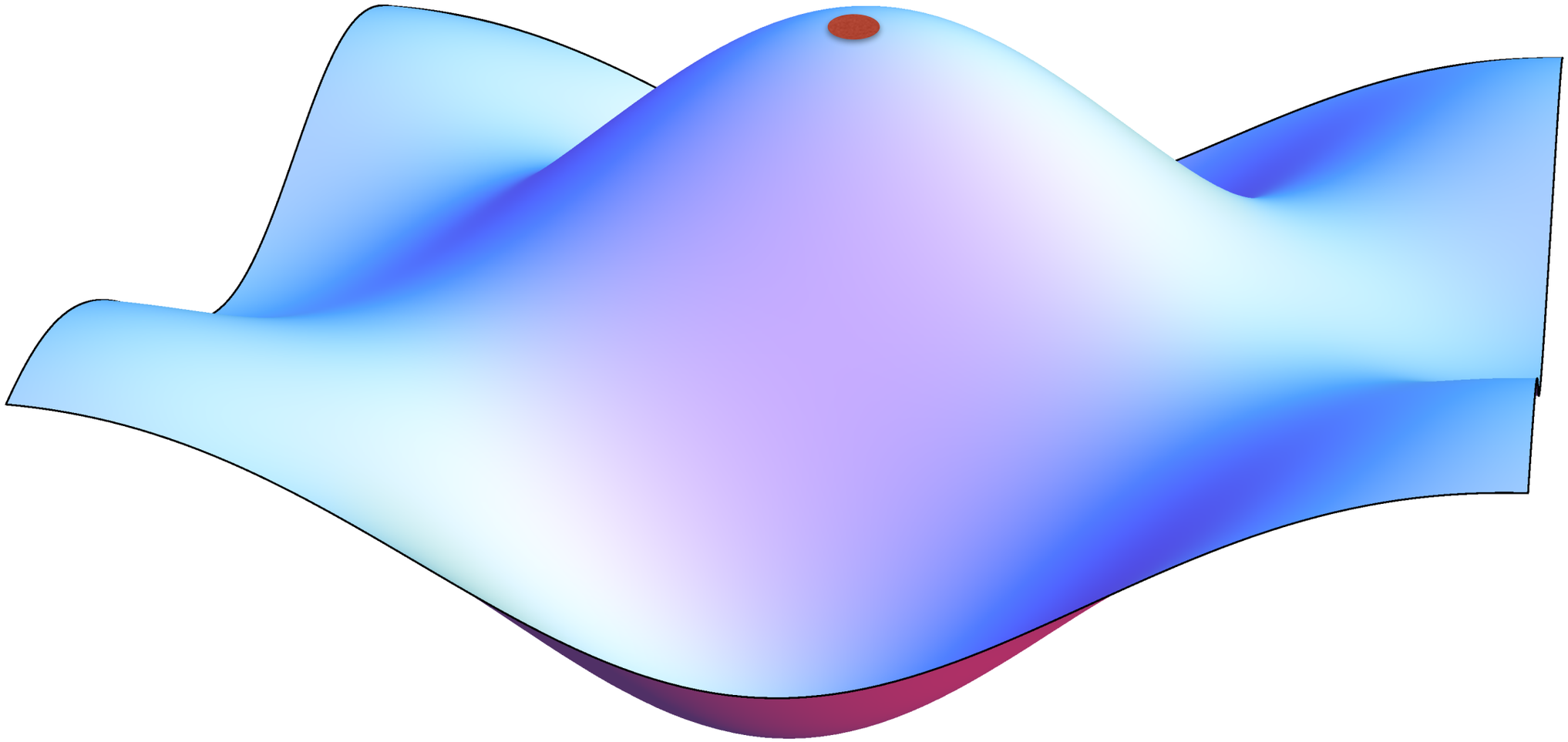
The Equivalence Principle



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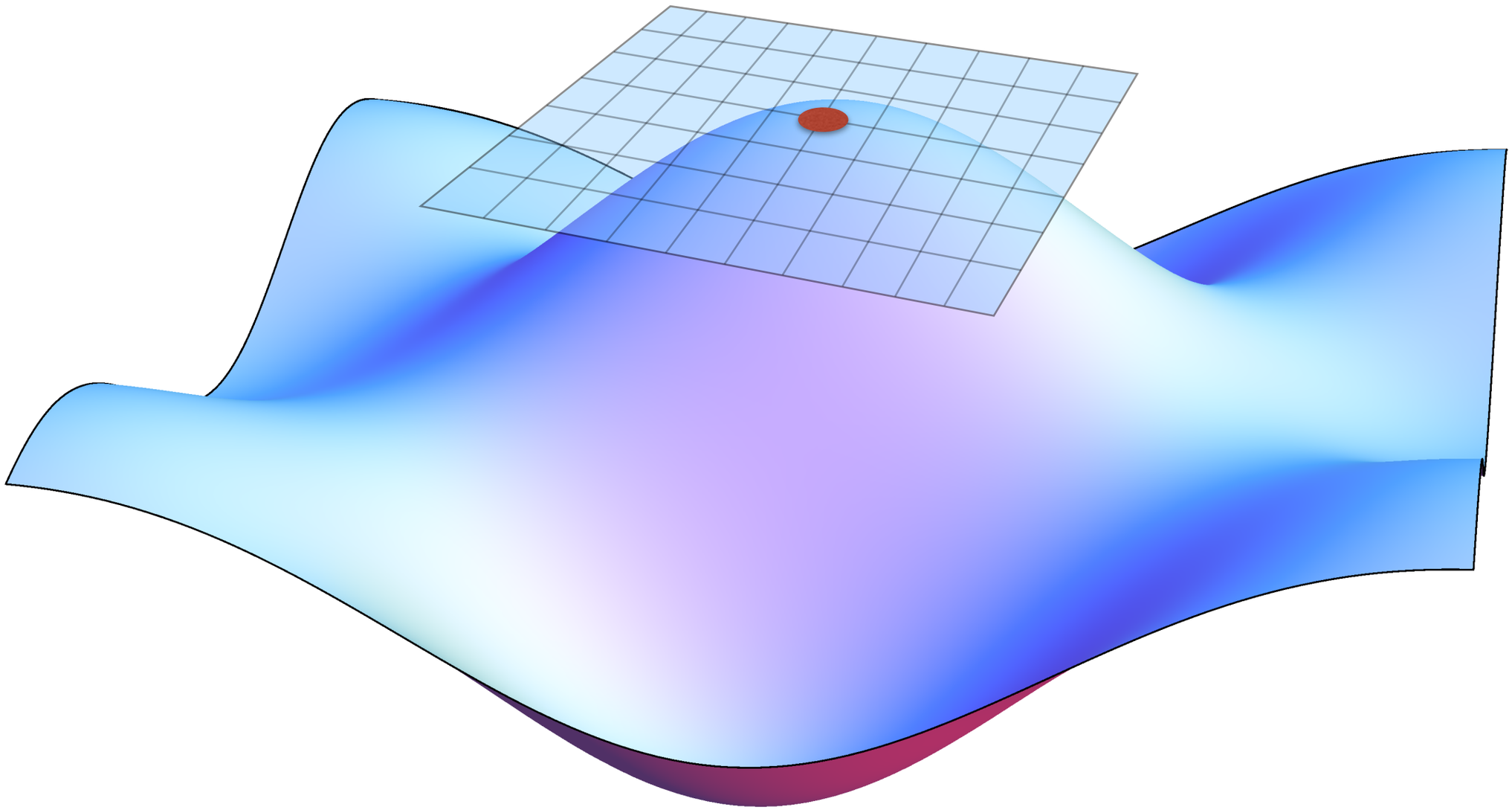

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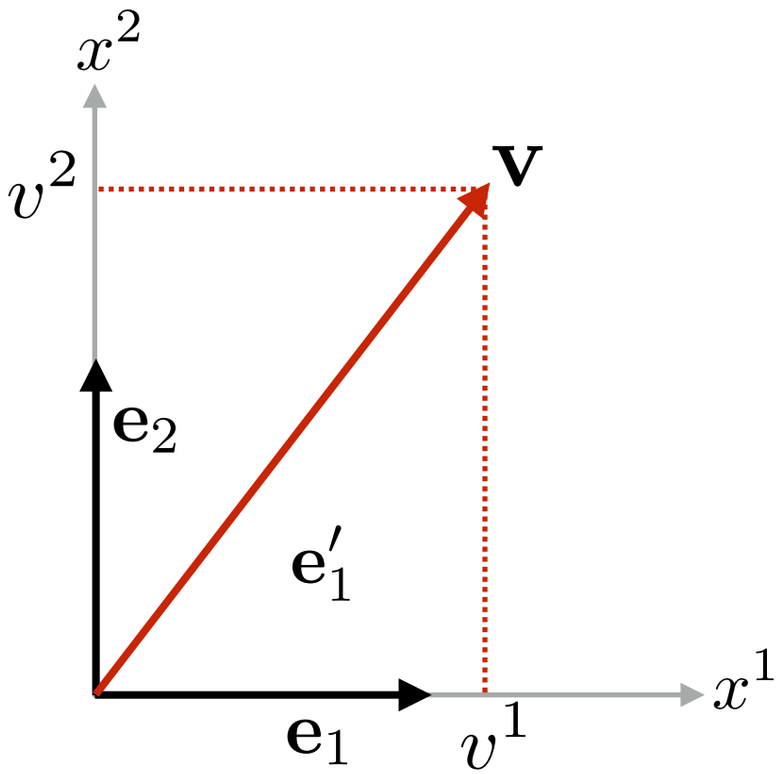
Mathematical Toolbox – Vectors

- **Vectors** are geometric entities – exist independent of any co-ordinate



Can be constructed from basis vectors (components)

$$\mathbf{v} = v^a \mathbf{e}_a = v'^a \mathbf{e}'_a$$

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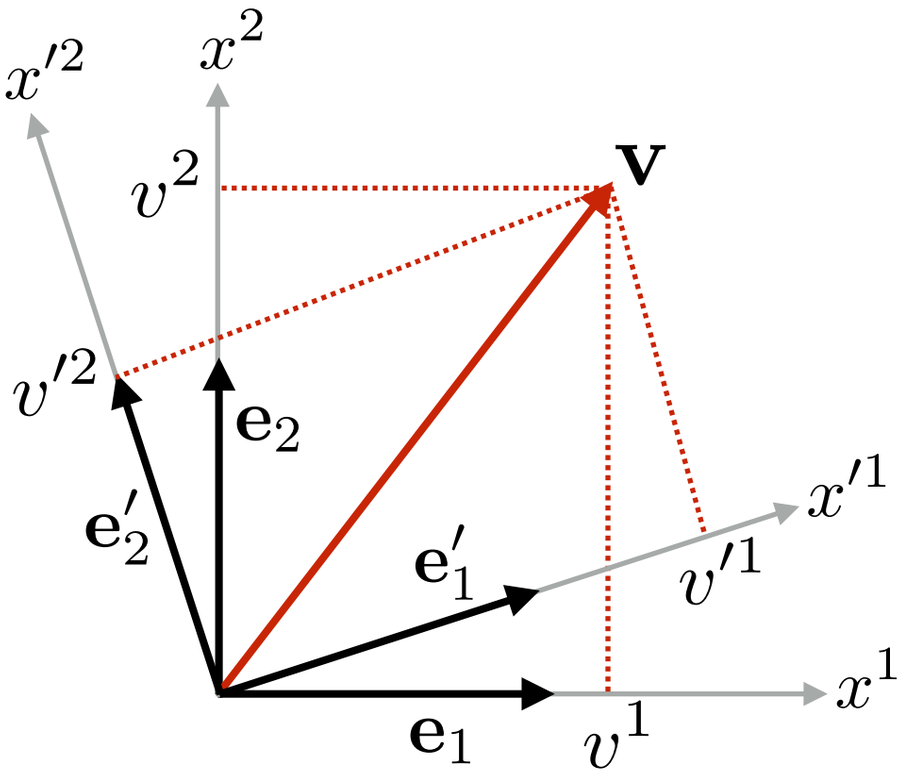

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Obey transformation law

$$v'^a = \left(\frac{\partial x'^a}{\partial x^b} \right) v^b$$

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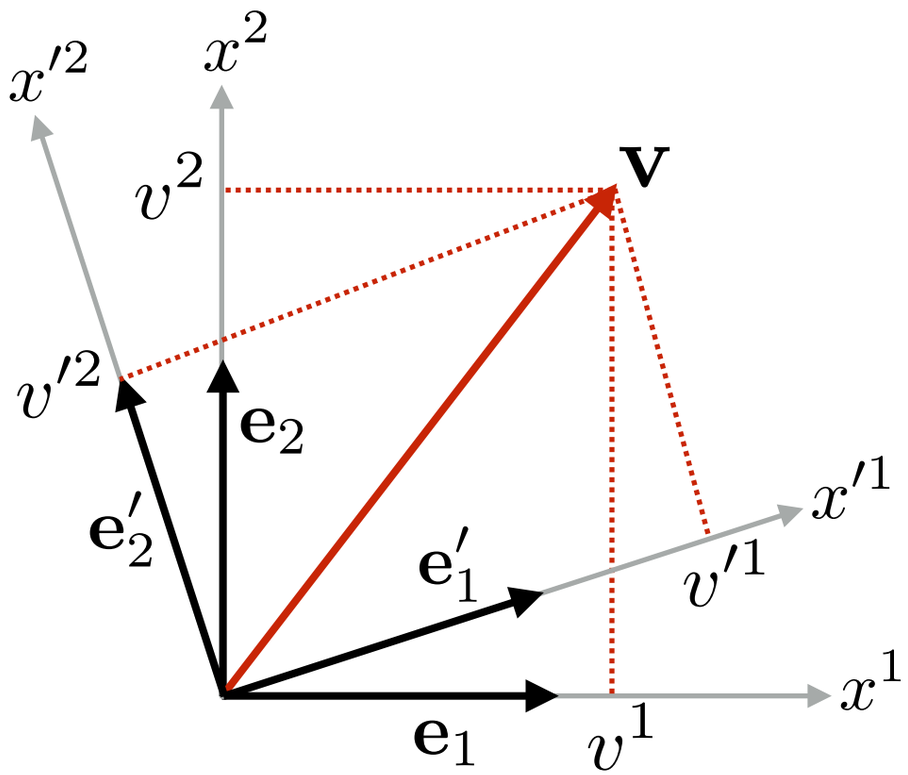

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- **Tensors** are also geometric entities – generalisation of vectors to higher rank and transform as $t'^{ab} = \left(\frac{\partial x'^a}{\partial x^c} \right) \left(\frac{\partial x'^b}{\partial x^d} \right) t^{cd}$

- **Scalars** have the same value in all frames

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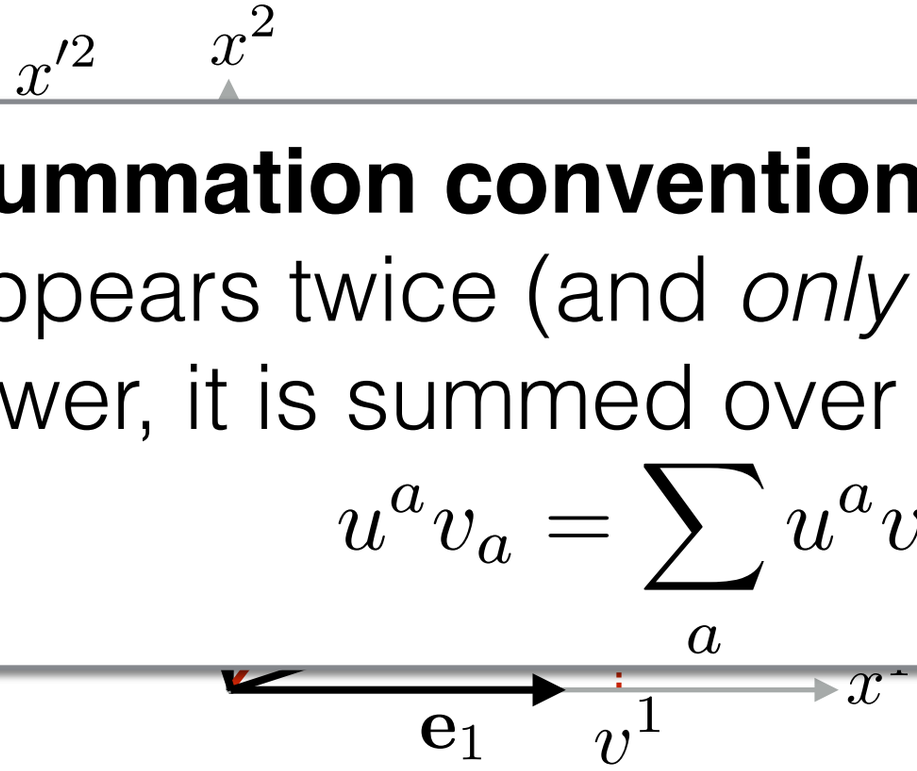
Mathematical Toolbox – Vectors

- **Vectors** are geometric entities – exist independent of any co-ordinate

Can be constructed from

Summation convention: Where a dummy index appears twice (and *only* twice), once upper and once lower, it is summed over

$$u^a v_a = \sum_a u^a v_a \qquad t^a_a = \sum_a t^a_a$$



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Some Useful 4-Vectors

- Position $\mathbf{x} = (ct, \vec{x}) = (ct, x, y, z)$
- Velocity $\mathbf{v} = \dfrac{d\mathbf{x}}{d\tau} = \left(c\dot{t}, \dot{\vec{x}}\right) = (\gamma c, \gamma v)$
- Momentum $\mathbf{p} = m_0\mathbf{v} = (\gamma mc, \gamma mv) = (E/c, \vec{p})$
- Wave vector $\mathbf{p} = \hbar\mathbf{k}$ $\mathbf{k} = \left(\omega/c, \vec{k}\right)$
- Current density $\mathbf{j} = \rho_c\mathbf{u}$ $\mathbf{j} = \left(\rho_c c, \vec{j}\right)$
- Electromagnetic potential $\mathbf{A} = \left(\phi/c, \vec{A}\right)$
- Proper time, τ, is a scalar

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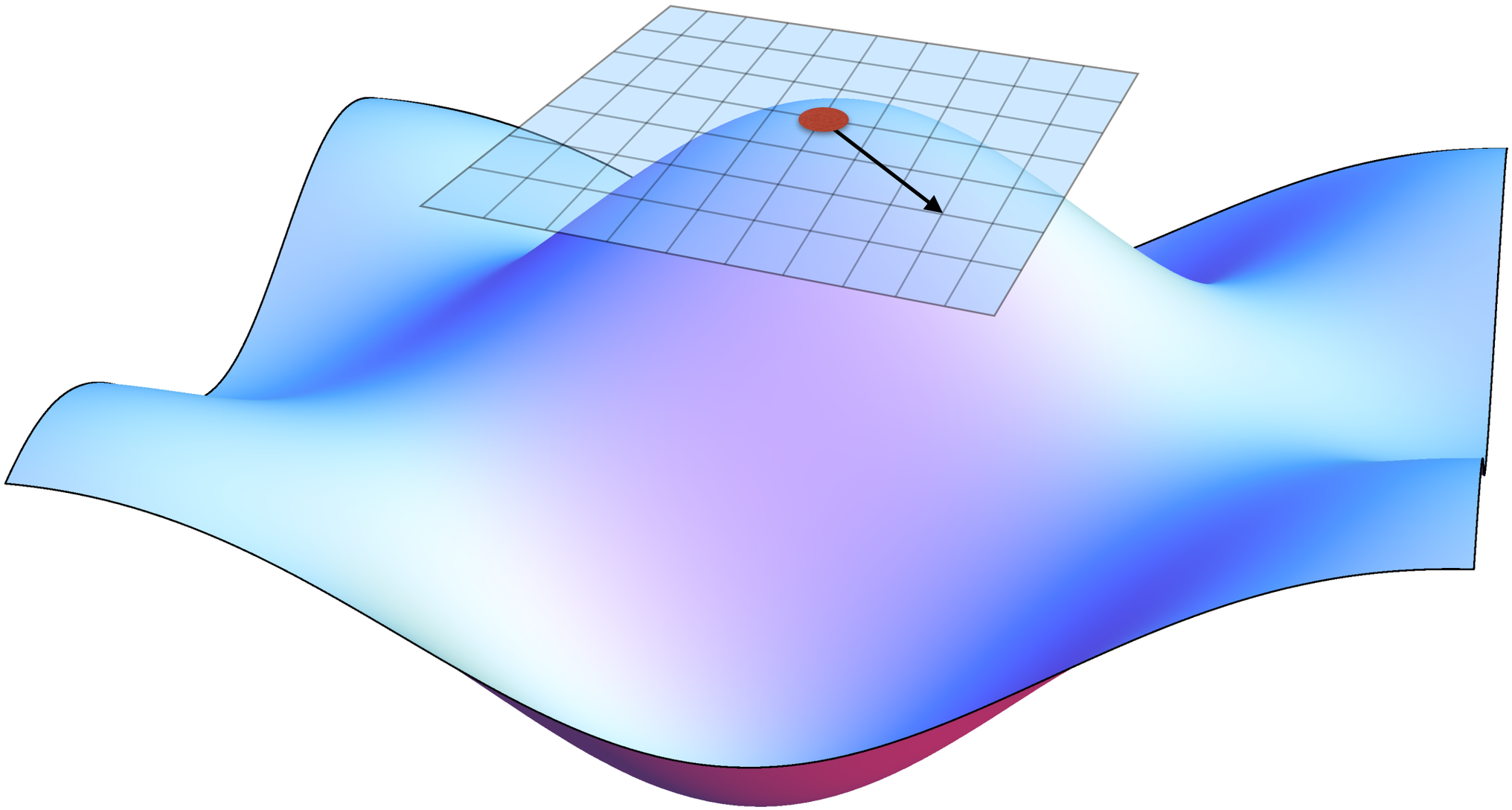
Vectors in Curved Spacetimes



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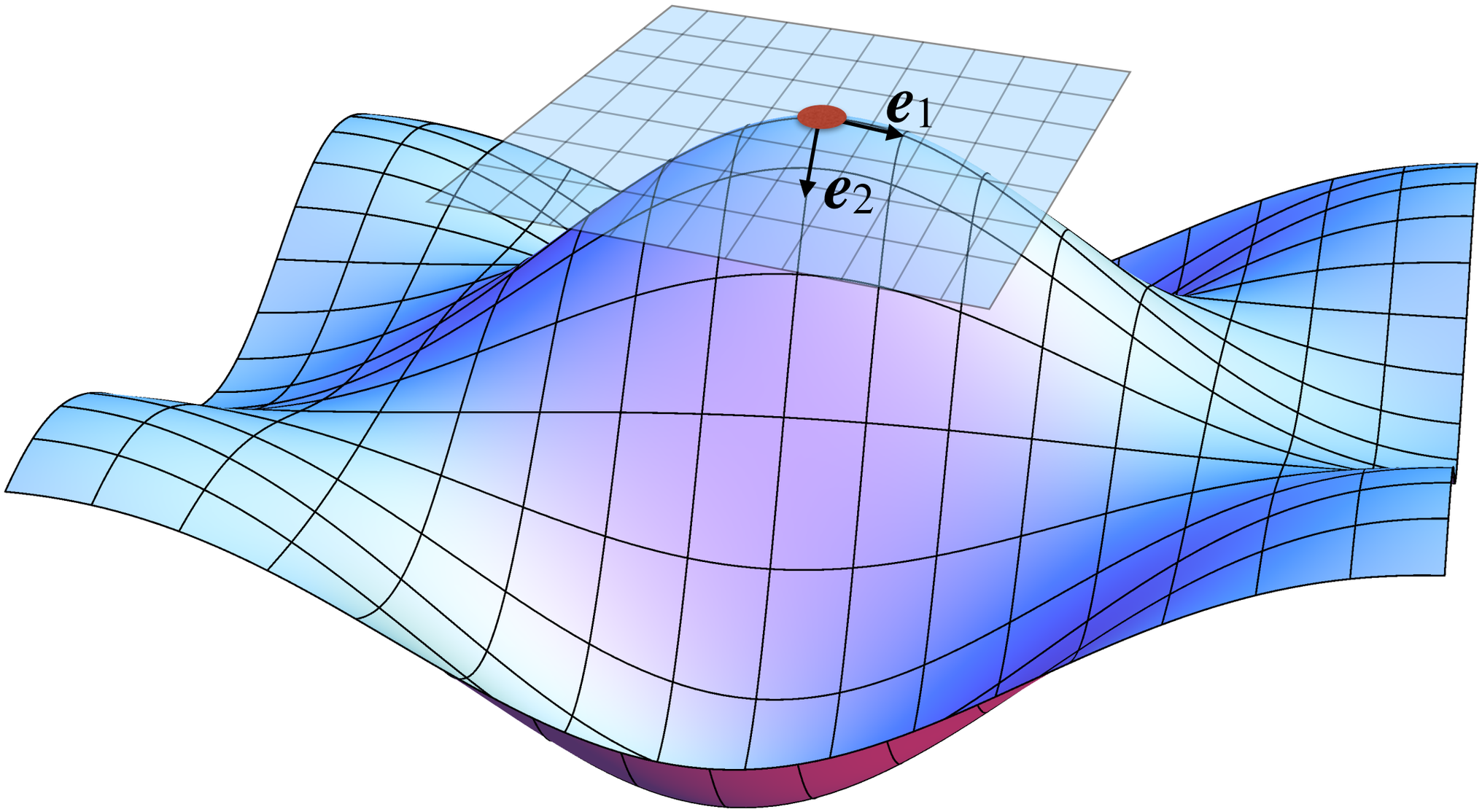
Co-ordinates and Basis Vectors



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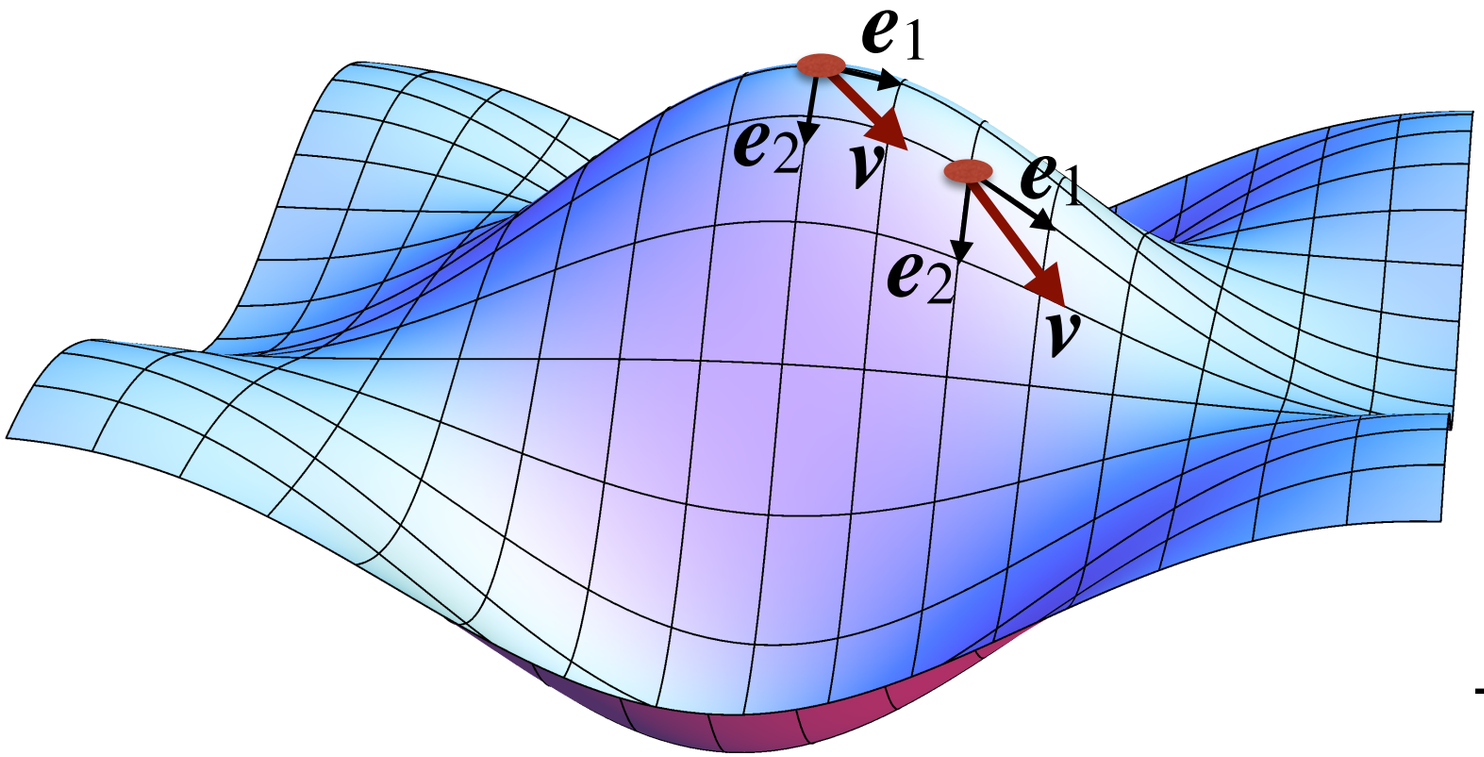
Differentiating Vectors



$$\frac{\partial \mathbf{v}}{dx^b} = \frac{\partial v^a}{\partial x^b}\mathbf{e}_a + v^a \frac{\partial \mathbf{e}_a}{\partial x^b}$$

$$\frac{\partial \mathbf{v}}{dx^b} = \underbrace{\left(\frac{\partial v^a}{\partial x^b} + v^c \Gamma^a_{cb}\right)}_{\nabla_b v^a} \mathbf{e}_a$$

The covariant derivative:

$$\nabla_b v^a = \frac{\partial v^a}{\partial x^b} + \Gamma^a_{cb} v^c$$

The connection: $\dfrac{\partial \mathbf{e}_a}{\partial x^b} = \Gamma^c_{ab}\mathbf{c}$

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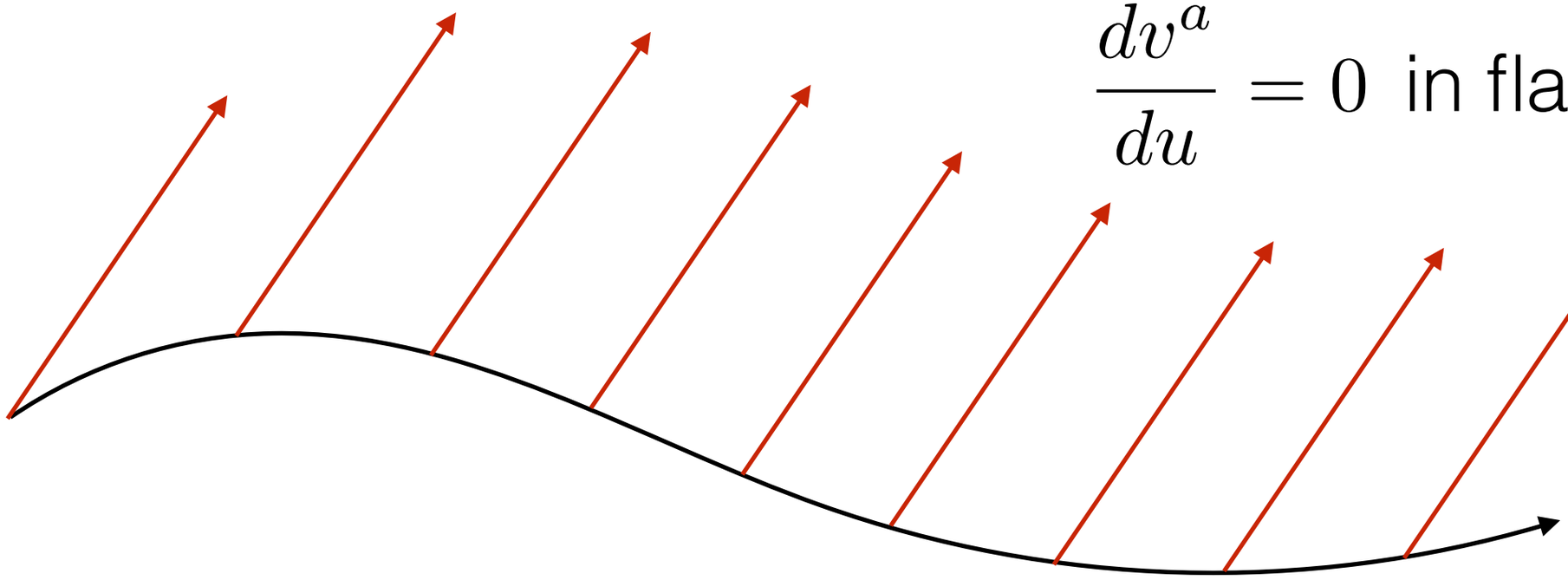
Parallel Transport



$$\frac{dv^a}{du} = 0 \text{ in flat space}$$

Vector parallel transported along a curve parameterised as $x^a(u)$

$$\frac{d\mathbf{v}}{du} = \frac{dv^a}{du}\mathbf{e}_a + v^a \frac{de_a}{du} = \frac{dv^a}{du}\mathbf{e}_a + v^a \frac{\partial \mathbf{e}_a}{\partial x^c}\frac{dx^c}{du}$$

$$\frac{d\mathbf{v}}{du} = \underbrace{\left(\frac{dv^a}{du} + v^a \Gamma^a_{bc} v^b \frac{dx^c}{du}\right)}_{\frac{Dv^a}{Du}} \mathbf{e}_a \qquad \frac{Dv^a}{Du} = 0 \text{ for parallel transport}$$

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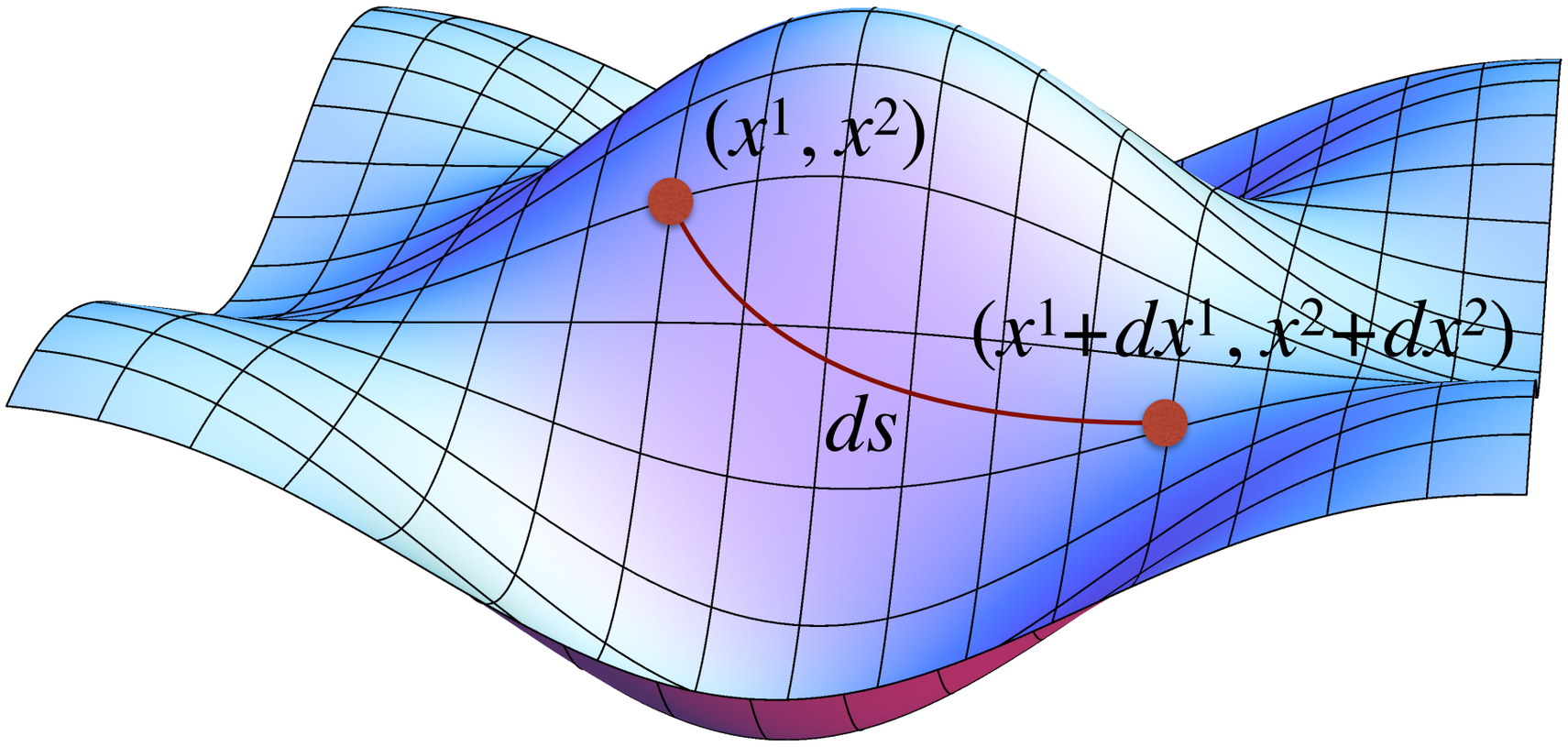
The Metric



Generalise the flat space line element $ds^2 = (dx^1)^2 + (dx^2)^2$

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + 2g_{01}(dx^1)(dx^2)$$

In general $\quad ds^2 = g_{ab}dx^a dx^b$

The metric tensor, g_{ab}, encodes all of the geometry of the spacetime, with $\mathbf{e}_a \cdot \mathbf{e}_b = g_{ab}$

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Contravariant vs. Covariant

- For any set of basis vectors $\{\mathbf{e}_a\}$, can define the *dual basis* $\{\mathbf{e}^a\}$ such that $\mathbf{e}_a \cdot \mathbf{e}^b = \delta_a^b$
- Can write a vector **v** in terms of either $\mathbf{v} = v^a \mathbf{e}_a$ $\mathbf{v} = v_a \mathbf{e}^a$
- v^a are the *contravariant* components of **v**
- v_a are the *covariant* components of **v**
- Since $\mathbf{e}_a \cdot \mathbf{e}_b = g_{ab}$, $v_a = g_{ab} v^b$ and $v^a = g^{ab} v_b$

$$[g^{ab}] = [g_{ab}]^{-1}$$

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Length of a Vector

- The line element is the length of the (infinitesimal) displacement vector

$$ds^2 = g_{ab} dx^a dx^b$$

- Apply to other vectors

$$|v|^2 = \mathbf{v} \cdot \mathbf{v} = g_{ab} v^a v^b = v^a v_a$$

- Dot products are invariant (same in all frames/co-ordinate systems)

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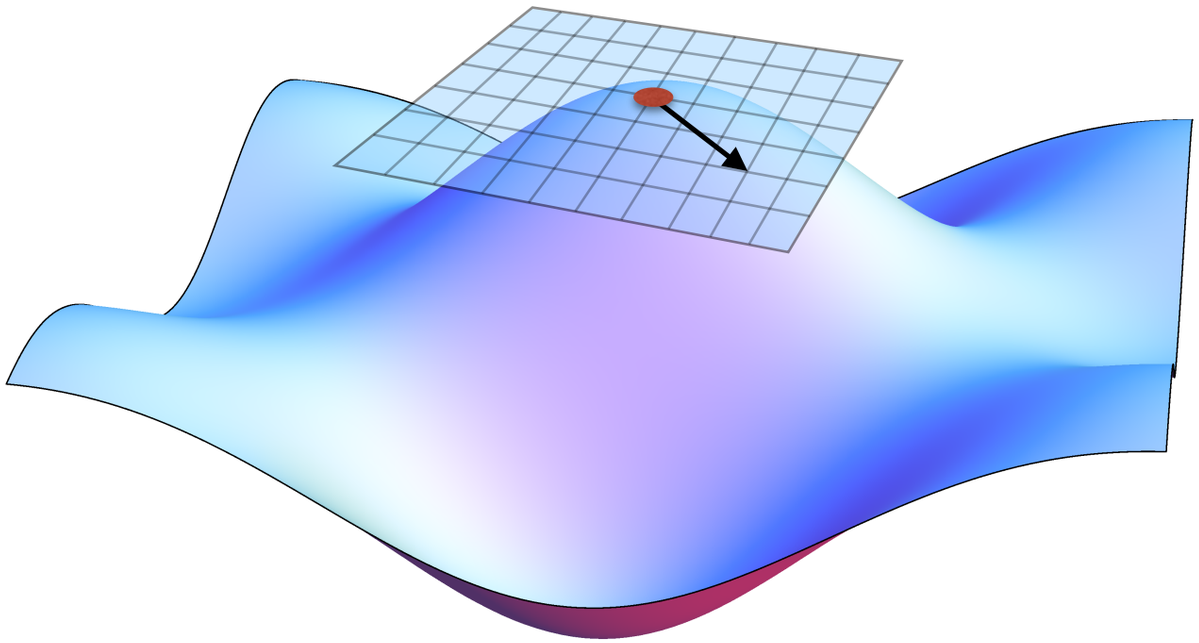
Making an Observation



e.g. 4-momentum **p** which has components in Minkowski space:

$$[p'^{\mu}] = (E', \vec{p}')$$

Observer measures energy

$$E' = \mathbf{p} \cdot \mathbf{e}'_{(t)} = \mathbf{p} \cdot \mathbf{u}$$

- Given a vector, ***v****,* an observer measures the components against their own set of basis vectors (flat, Minkowski space in their rest frame)
- They are at rest in their own frame, so their timelike basis vector must be parallel to their 4-velocity
- Can construct an orthogonal set of vectors around this to give *x, y* and *z*

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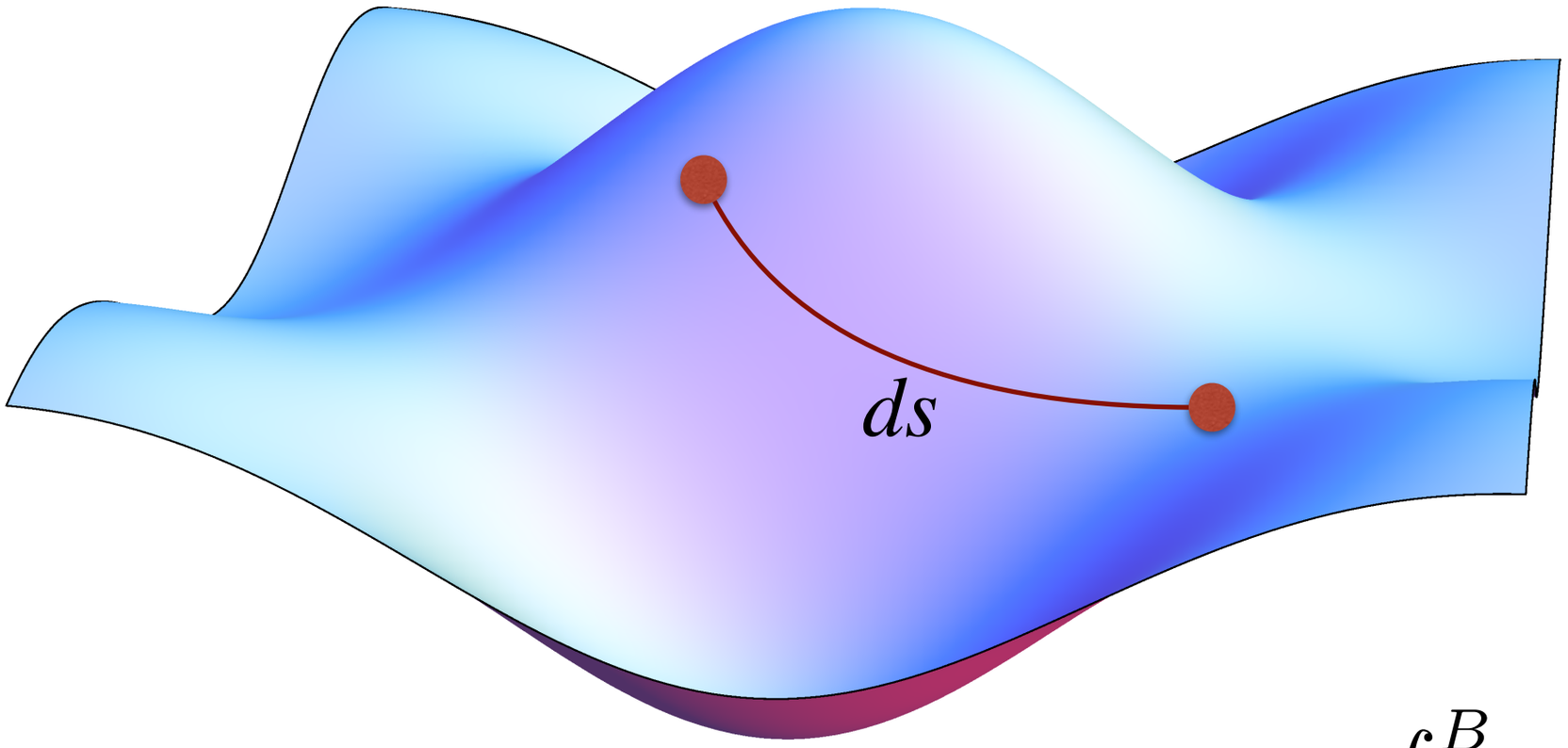
Geodesics



- Shortest path between 2 points $s = \int_A^B g_{ab}\dot{x}^a\dot{x}^b\,du$
- Tangent vector (the momentum) is parallel-transported along the geodesic

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0$$

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What is the appropriate Lorentz transformation for the density of a cube of matter?

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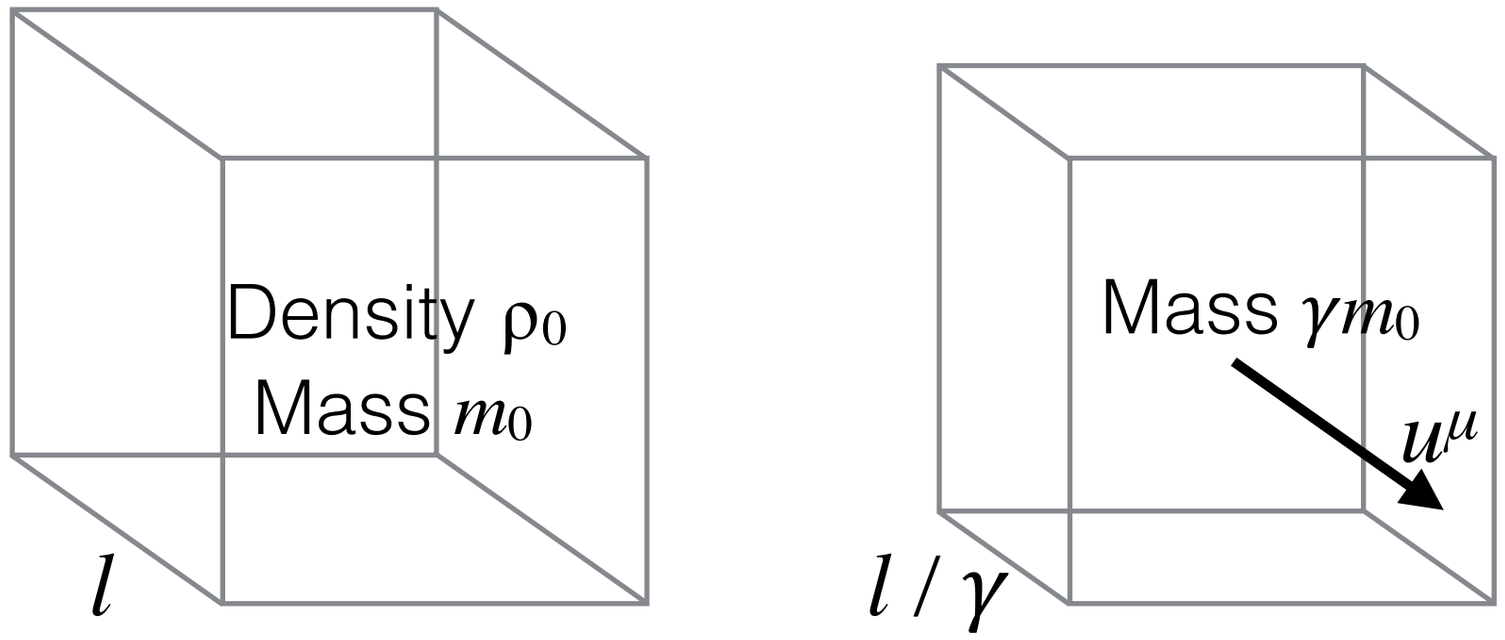
Describing the Matter



- Density increases by factor γ^2 – transforms as a rank 2 tensor

$$T^{\mu\nu} = \rho_0 u^\mu u^\nu$$

- The energy-momentum tensor for a perfect fluid

$$T^{\mu\nu} = \left(\rho_0 + p/c^2\right) u^\mu u^\nu - p g^{\mu\nu}$$

- To conserve energy and momentum $\nabla_\mu T^{\mu\nu} = 0$

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Describing the Curvature

- In flat space, it doesn't matter which order you perform a covariant derivative (because in flat space these are just partial derivatives) $\nabla_c \nabla_b v^a = \nabla_b \nabla_c v^a$

- Can measure the curvature of space by changing the order of covariant derivatives

$$\nabla_c \nabla_b v_a - \nabla_b \nabla_c v_a = R^d_{abc} v_d$$

- $R^a{}_{bcd}$ is the rank 4 Riemann tensor (or curvature tensor)

- Can also contract it (sum over indices) to give the Ricci tensor (rank 2) and scalar

$$R_{ab} = R^c_{abc} \qquad\qquad R = R^a_a$$

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The Einstein Field Equations

- Curvature of spacetime is related to the matter content and energy-momentum tensor is rank 2

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

- Needs to reduce to the Newtonian limit (terms no higher than linear in second derivatives of metric)

- T is symmetric so G must be symmetric

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

Einstein 1916

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The Einstein Field Equations

- Curvature of spacetime is related to the matter content and energy-momentum tensor is rank 2

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- Needs to reduce to the Newtonian limit (terms no higher than linear in second derivatives of metric)

- T is symmetric so G must be symmetric

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

Einstein 1916

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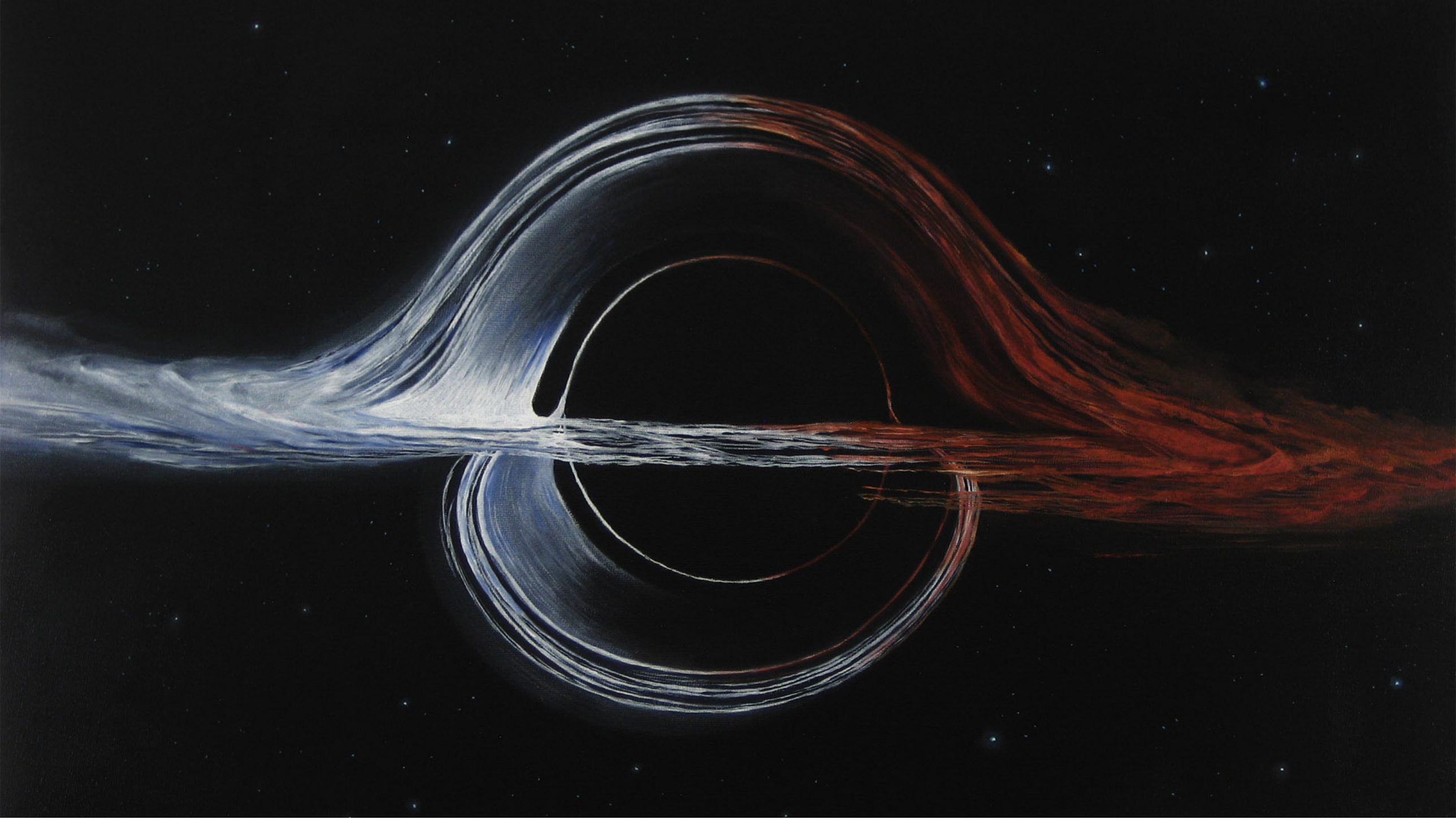

Describing the Black Hole

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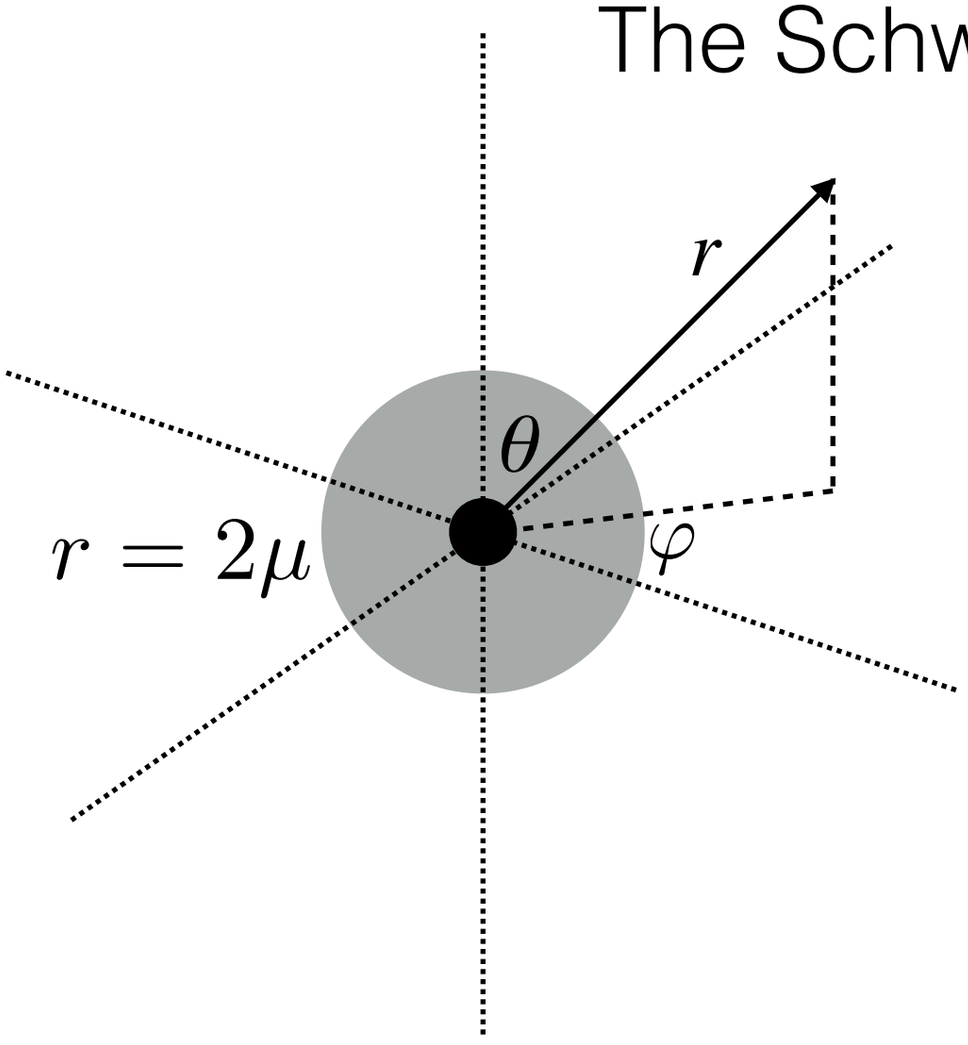
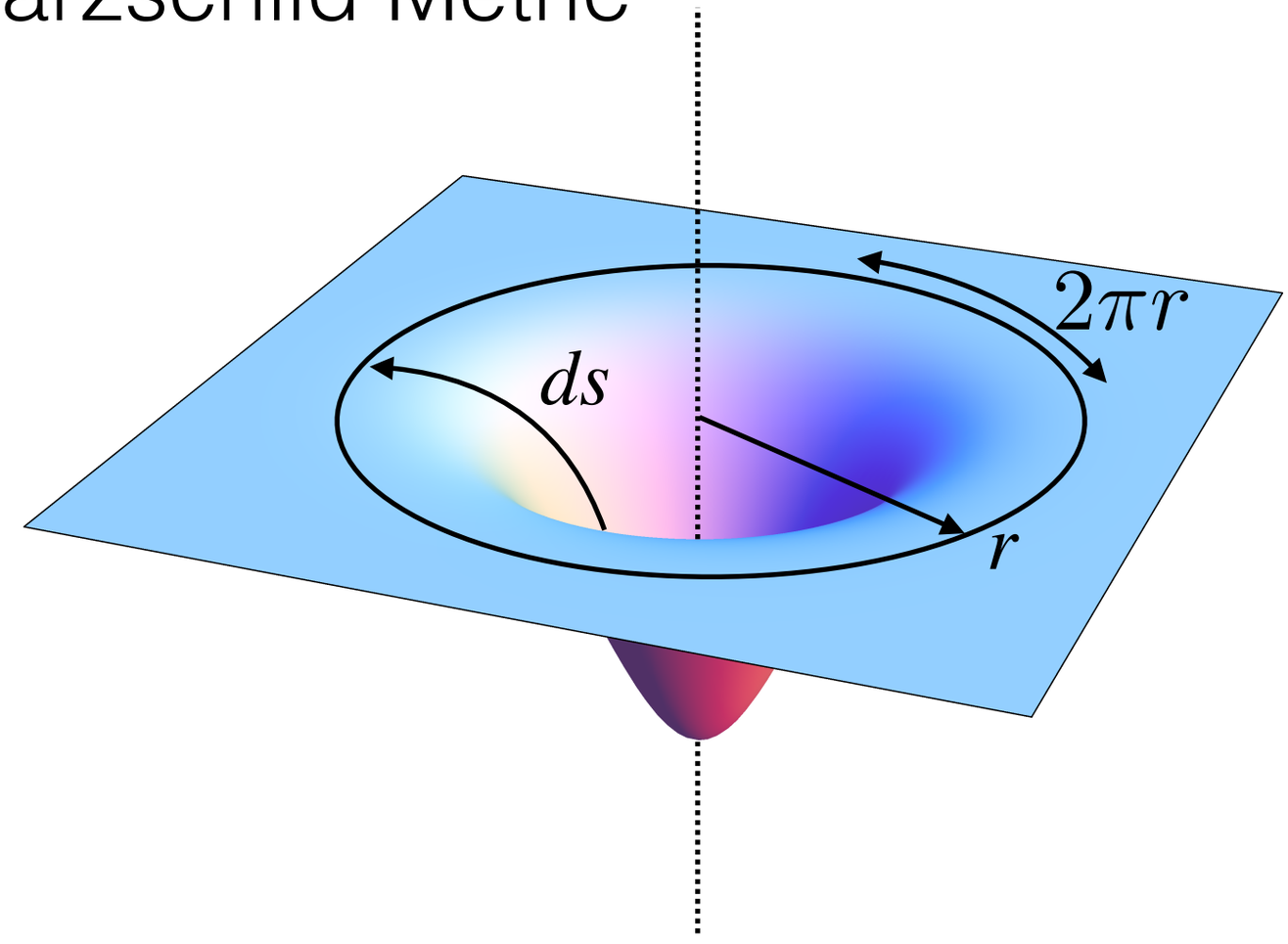
Gravity around a Point Mass

The Schwarzschild Metric





$$ds^2 = c^2\left(1 - \frac{2\mu}{r}\right)dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi\right)$$

where $\mu = \dfrac{GM}{c^2}$

Schwarzschild 1916

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Gravitational Redshift

- In an observer's rest frame, spacetime is described by the Minkowski metric (using their co-ordinates)

- Their own worldline $ds^2 = c^2 d\tau^2$

$$c^2 d\tau^2 = c^2 \left(1 - \frac{2\mu}{r}\right) dt^2$$

- For observers closer to the black hole, proper time elapses more slowly

- Photons climbing out of the potential are redshifted

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Geodesic Equations

In the equatorial plane of the Schwarzschild geometry ($\theta = \pi / 2$)

$$\left(1 - \frac{2\mu}{r}\right) \dot{t} = k \tag{1}$$

$$c^2 \left(1 - \frac{2\mu}{r}\right) \dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\varphi}^2 = \begin{cases} c^2, & \text{massive particles} \\ 0, & \text{photons} \end{cases} \tag{2}$$

$$r^2 \dot{\varphi} = h \tag{3}$$

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Constants of Motion

$$\left(1 - \frac{2\mu}{r}\right)\dot{t} = k \qquad\qquad r^2\dot{\varphi} = h$$

- *h* is identified with the angular momentum
- To understand *k*, consider the 4-momentum of the particle
- An observer with 4-velocity u measures the components of the 4-momentum as $[p^\mu] = (E, \vec{p})$
- Energy is measured as, $E = \mathbf{p} \cdot \mathbf{u}$ which for an observer at rest at infinity

$$E = g_{0\mu}p^\mu = p_0 = kc^2$$

- *k* is the identified with the energy of the particle in its orbit $k = \dfrac{E}{m_0 c^2}$

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Energy and Shape Equations

- Energy equation — substitute (1) and (3) into (2)

$$\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{2GM}{r} = c^2(k^2 - 1)$$

- Shape equation — substitute (3) into above and differentiate ($u = 1 / r$)

$$\frac{d^2u}{d\varphi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2$$

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Radial Motion

Setting $d\theta = d\varphi = 0$
energy equation becomes

$$\dot{r}^2 = c^2(k^2 - 1) + \frac{GM}{r^2}$$

$$\ddot{r} = -\frac{GM}{r^2}$$

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Radial Motion

Setting $d\theta = d\varphi = 0$
energy equation becomes

$$\dot{r}^2 = c^2(k^2 - 1) + \frac{GM}{r^2}$$

$$\ddot{r} = -\frac{GM}{r^2}$$

$$\frac{dt}{d\tau} = \left(1 - \frac{2\mu}{r}\right)^{-1}$$

$$\frac{dr}{d\tau} = \left(\frac{2\mu c^2}{r}\right)^{\frac{1}{2}}$$

$$\tau \to \frac{2}{3}\sqrt{\frac{r_0^3}{2\mu c^2}} \quad \text{as} \quad r \to 0$$

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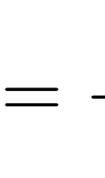

$$\frac{dt}{d\tau} = \left(1 - \frac{2\mu}{r}\right)^{-1}$$

$$\frac{dr}{d\tau} = \left(\frac{2\mu c^2}{r}\right)^{\frac{1}{2}}$$

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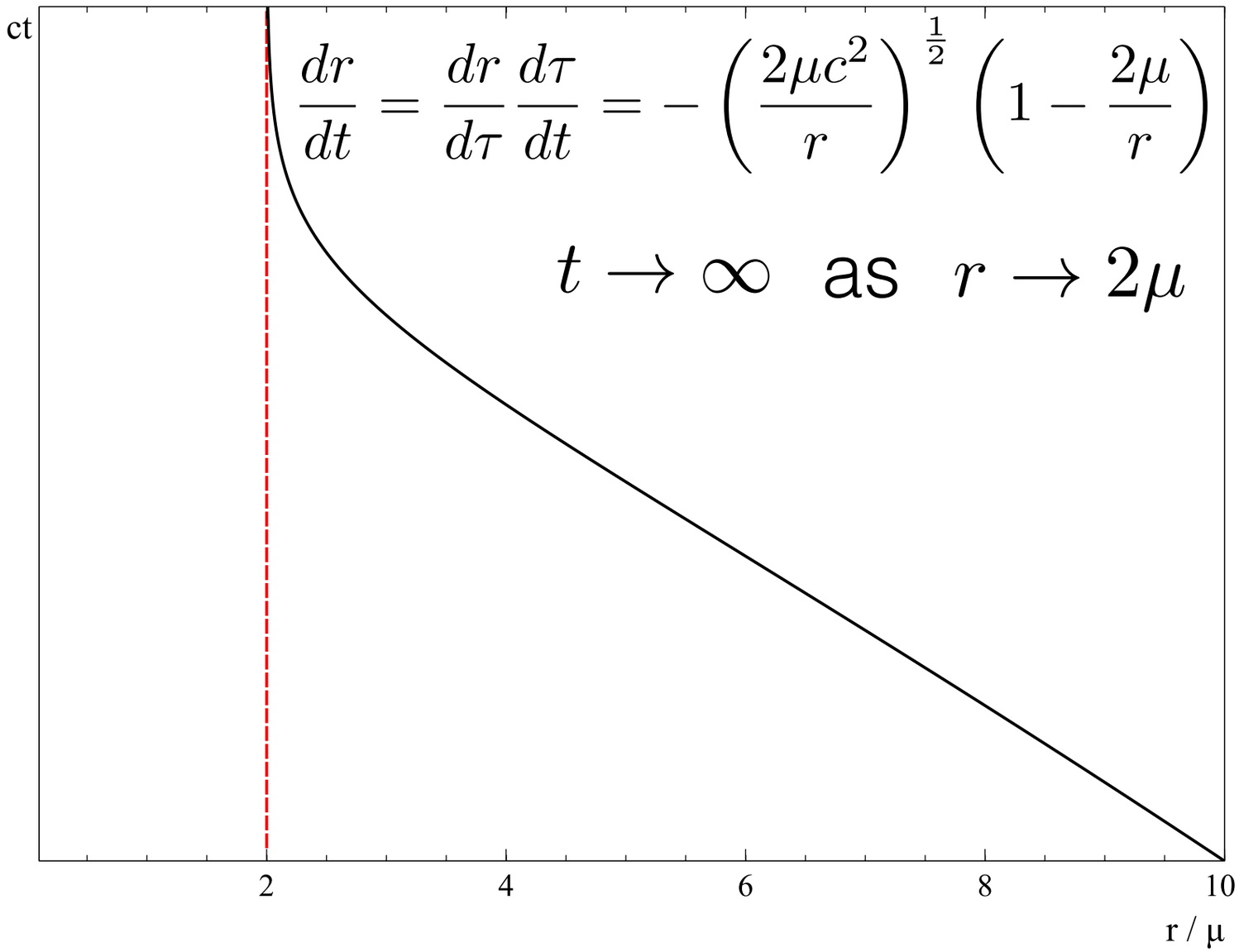

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If you were to drop objects into a black hole, what would it look like some time later?

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Orbits

- Set $dr = d\theta = 0$ for orbits in the equatorial plane
- Energy equation $\frac{h^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{2GM}{r} = c^2(k^2 - 1)$
- Shape equation $u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2$

$$h^2 = \frac{\mu c^2 r^2}{r - 3\mu} \qquad k = \frac{1 - 2\mu/r}{(1 - 3\mu/r)^{\frac{1}{2}}}$$

$$\frac{d\varphi}{dt} = \sqrt{\frac{GM}{r^3}}$$

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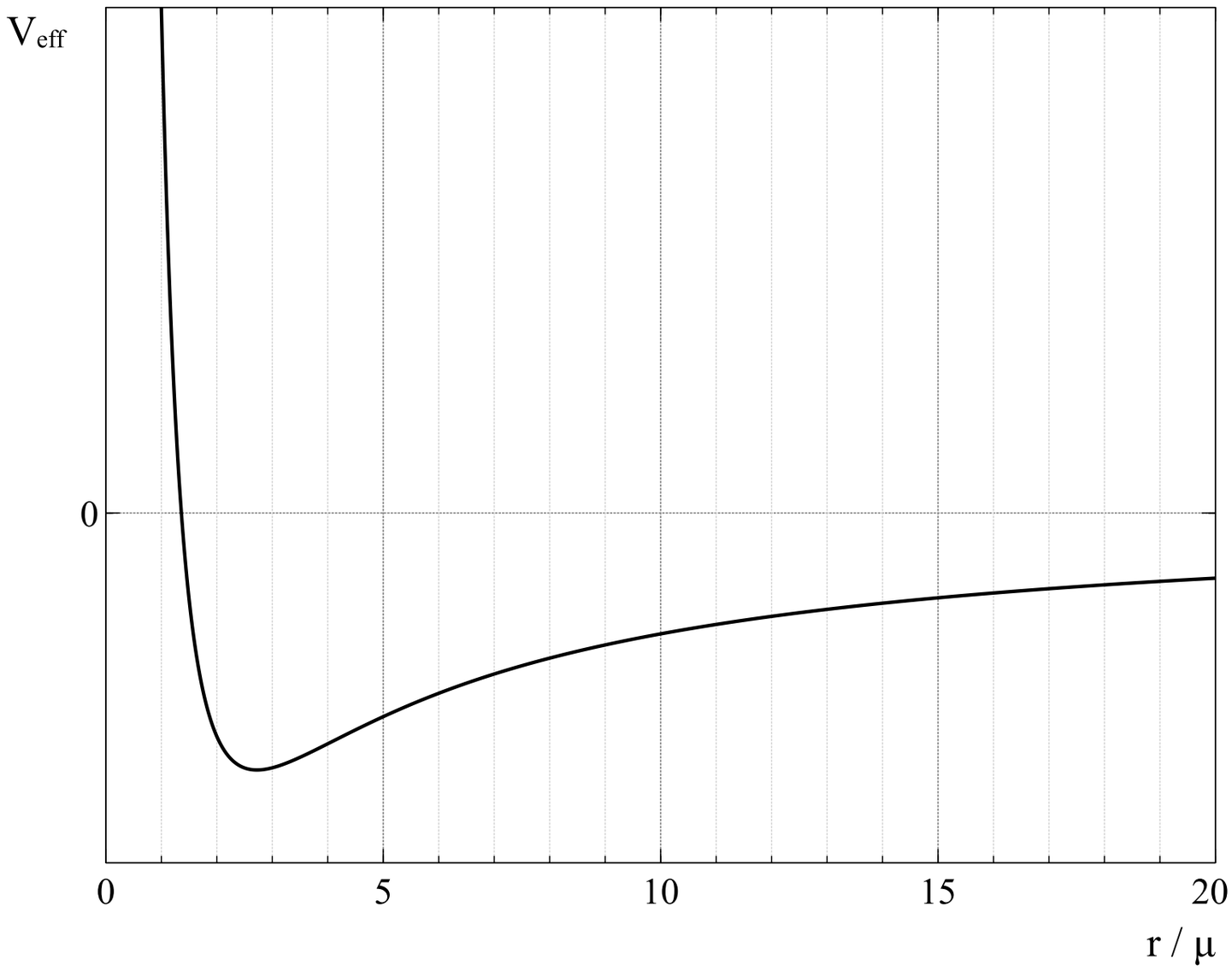
Stability of Orbits



Newtonian orbits

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 \underbrace{-\frac{GM}{r}+\frac{h^2}{2r^2}}_{V_{\mathrm{eff}}} = E$$

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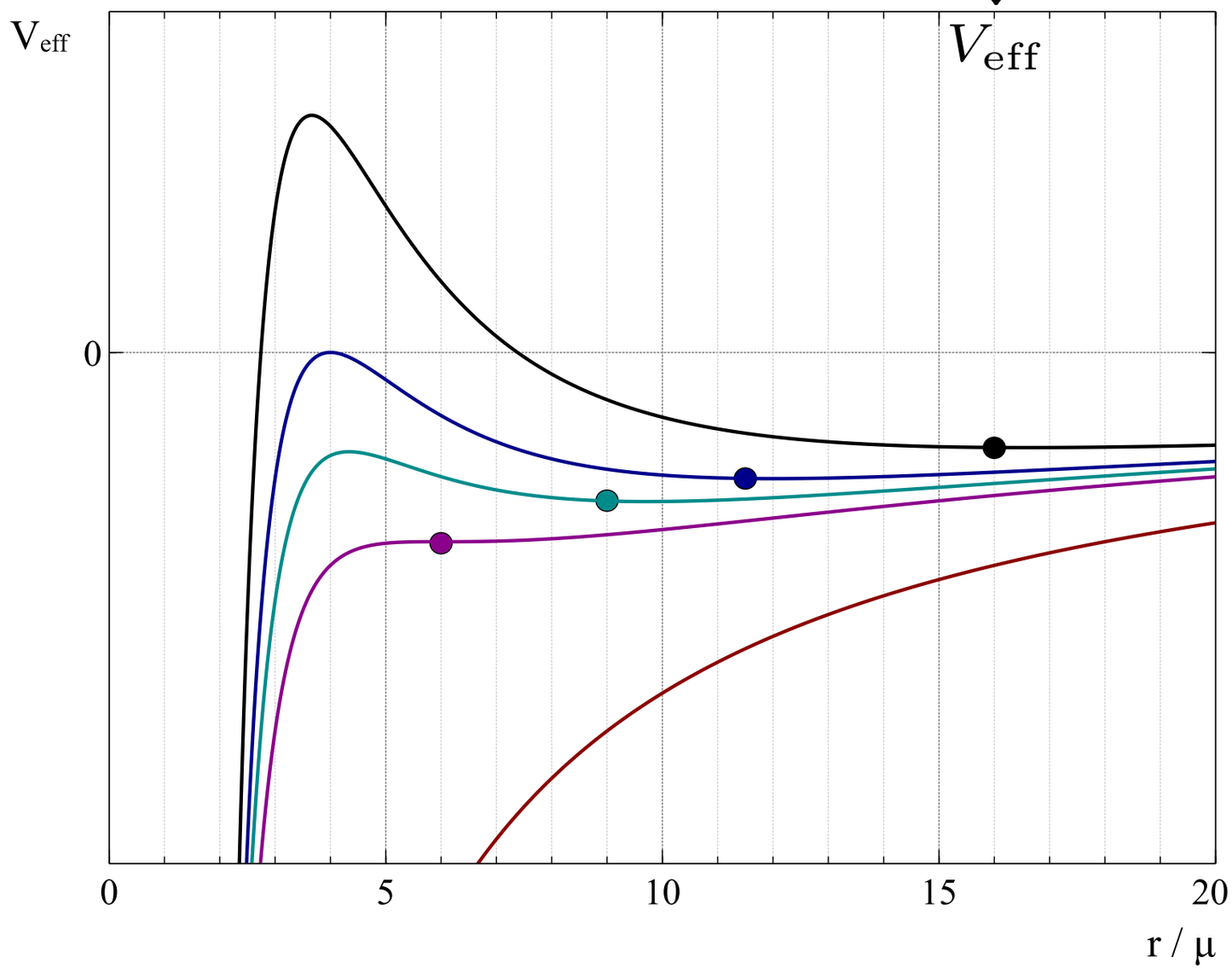
Stability of Orbits

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 \underbrace{-\frac{\mu c^2}{r} + \frac{h^2}{2r^2} - \frac{\mu h^2}{r^3}}_{V_{\text{eff}}} = \frac{c^2}{2}\left(k^2 - 1\right)$$



In Schwarzschild spacetime

Innermost stable circular orbit (ISCO) at $r = 6\mu$

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(In simple terms) what would an accretion disc look like around a black hole?

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Motion of Photons

$$c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2-\left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2-r^2\dot{\varphi}^2=\begin{cases}c^2, & \text{massive particles}\\ 0, & \text{photons}\end{cases}$$

- Energy equation becomes

$$\dot{r}^2+\frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right)=c^2k^2$$

- And the shape equation

$$\frac{d^2u}{d\varphi^2}+u=\frac{3GM}{c^2}u^2$$

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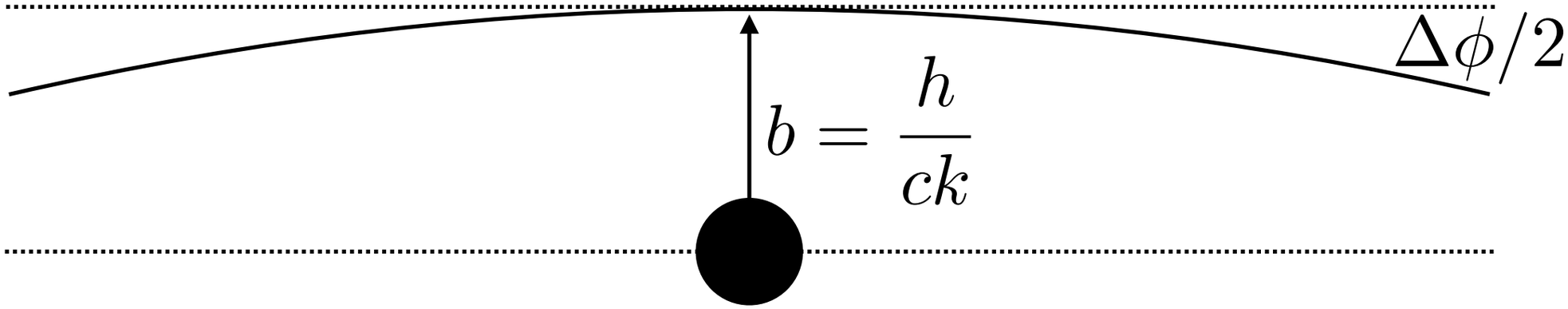
Gravitational Lensing



From the shape equation for a small perturbation Δu to the straight line path

$$\frac{d^2u}{d\varphi^2} + u = \frac{3GM}{c^2}u^2 \qquad u = \frac{\sin\varphi}{b} + \Delta u$$

$$r \to \infty \qquad \Delta\phi = \frac{4GM}{c^2 b}$$

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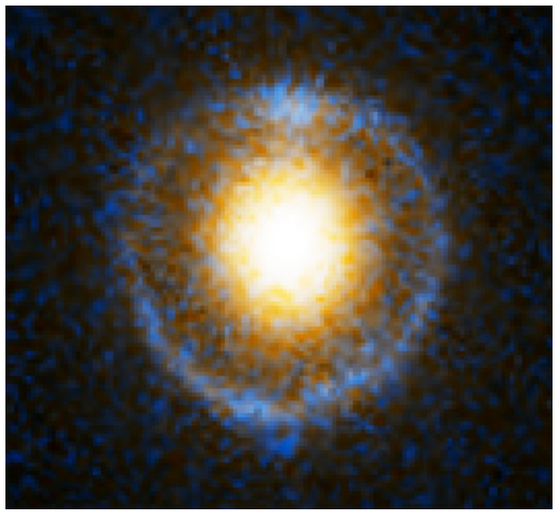
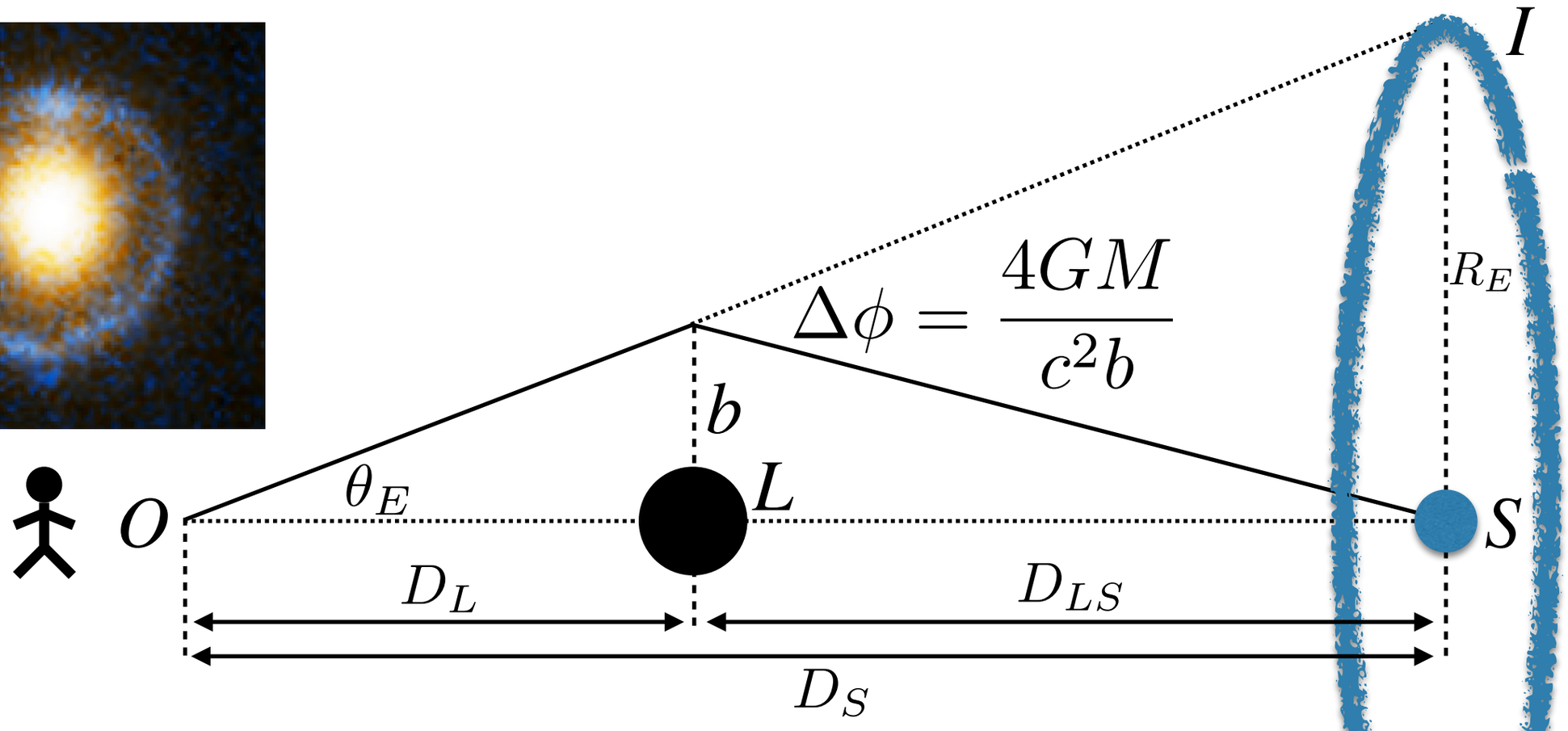
Gravitational Lensing





$$R_E = \Delta\phi D_{LS} \qquad b = R_E \frac{D_L}{D_S}$$

$$b = \sqrt{\frac{4GM}{c^2} D_{LS} \frac{D_L}{D_S}} \qquad \theta_E = \frac{R_E}{D_S} = \sqrt{\frac{4GM}{c^2} \frac{D_{LS}}{D_L D_S}}$$

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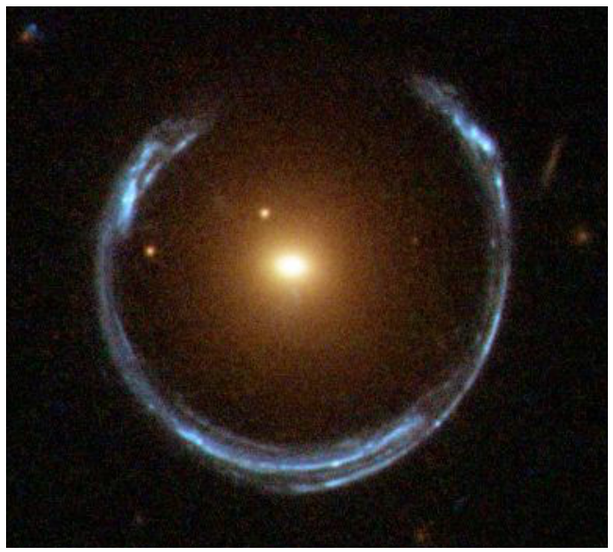
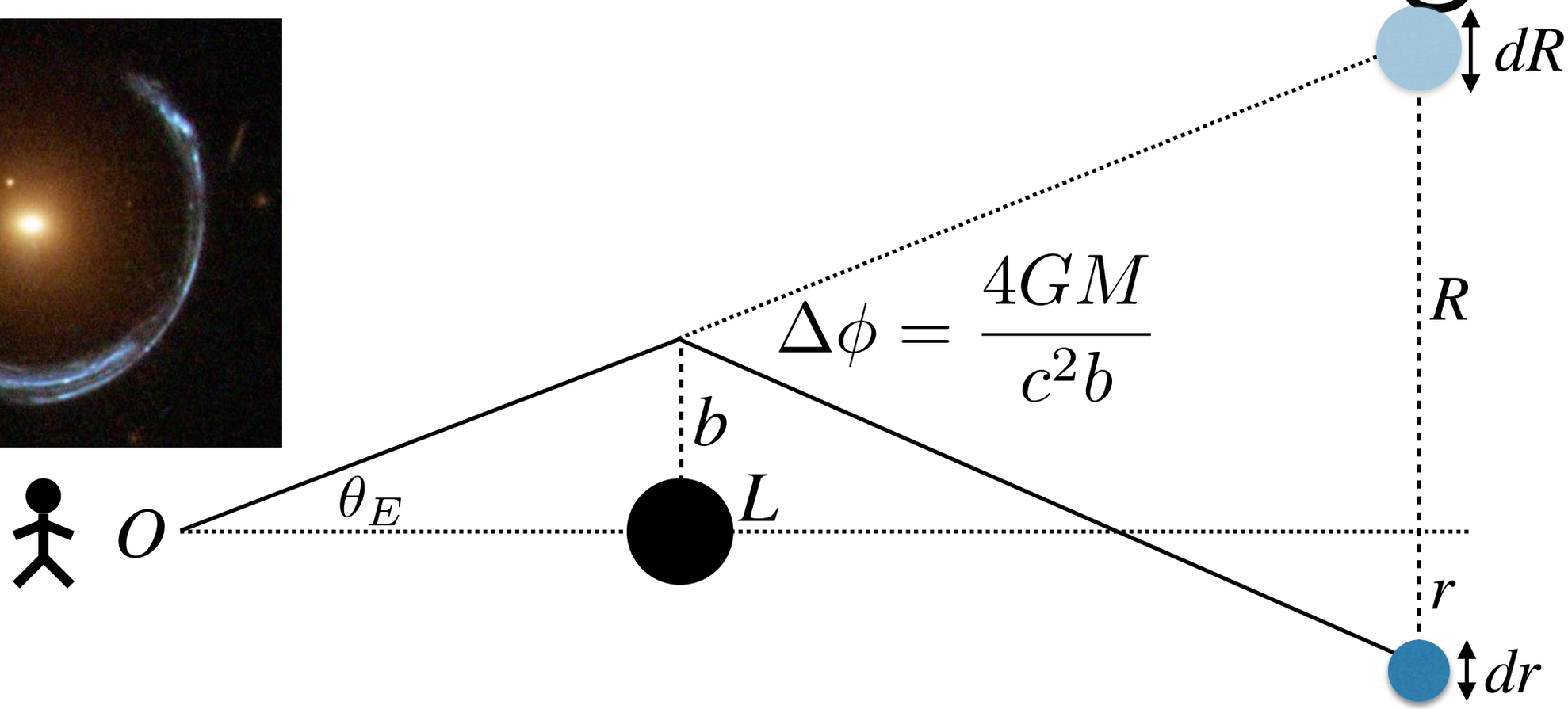
Gravitational Lensing





$$R = \frac{1}{2}\left(\pm\sqrt{r^2 + 4R_E^2} - r\right)$$

$$\text{Magnification} = \left|\frac{R}{r}\frac{dR}{dr}\right| = \frac{\rho^2 + 2}{\rho\sqrt{\rho^2 + 4}} \qquad \rho = \frac{r}{R_E}$$

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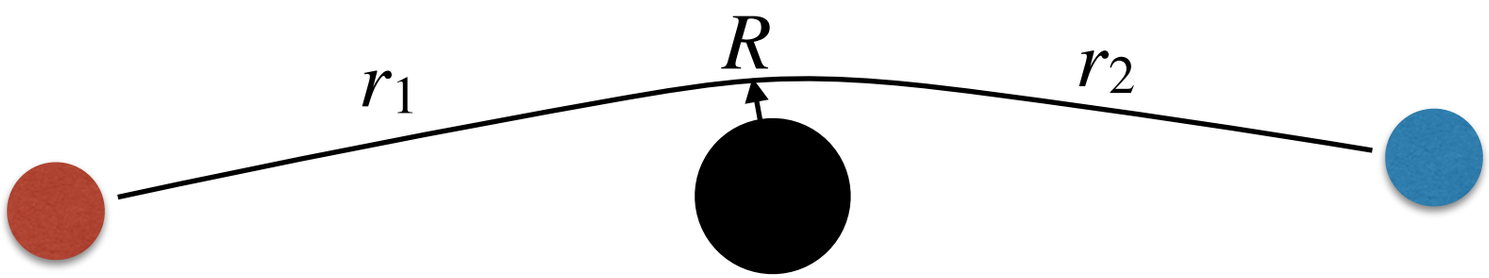
Light Travel Delays



$$ct = r - r_0 + 2\mu \ln \left| \frac{r - 2\mu}{r_0 - 2\mu} \right|$$

For round trip

$$\Delta(ct) = 4\mu \ln \left| \frac{(r_1 - 2\mu)(r_2 - 2\mu)}{(R - 2\mu)^2} \right| \sim 4\mu \ln \left| \frac{r_1 r_2}{R^2} \right|$$

~200μs for echoes between Earth and Venus

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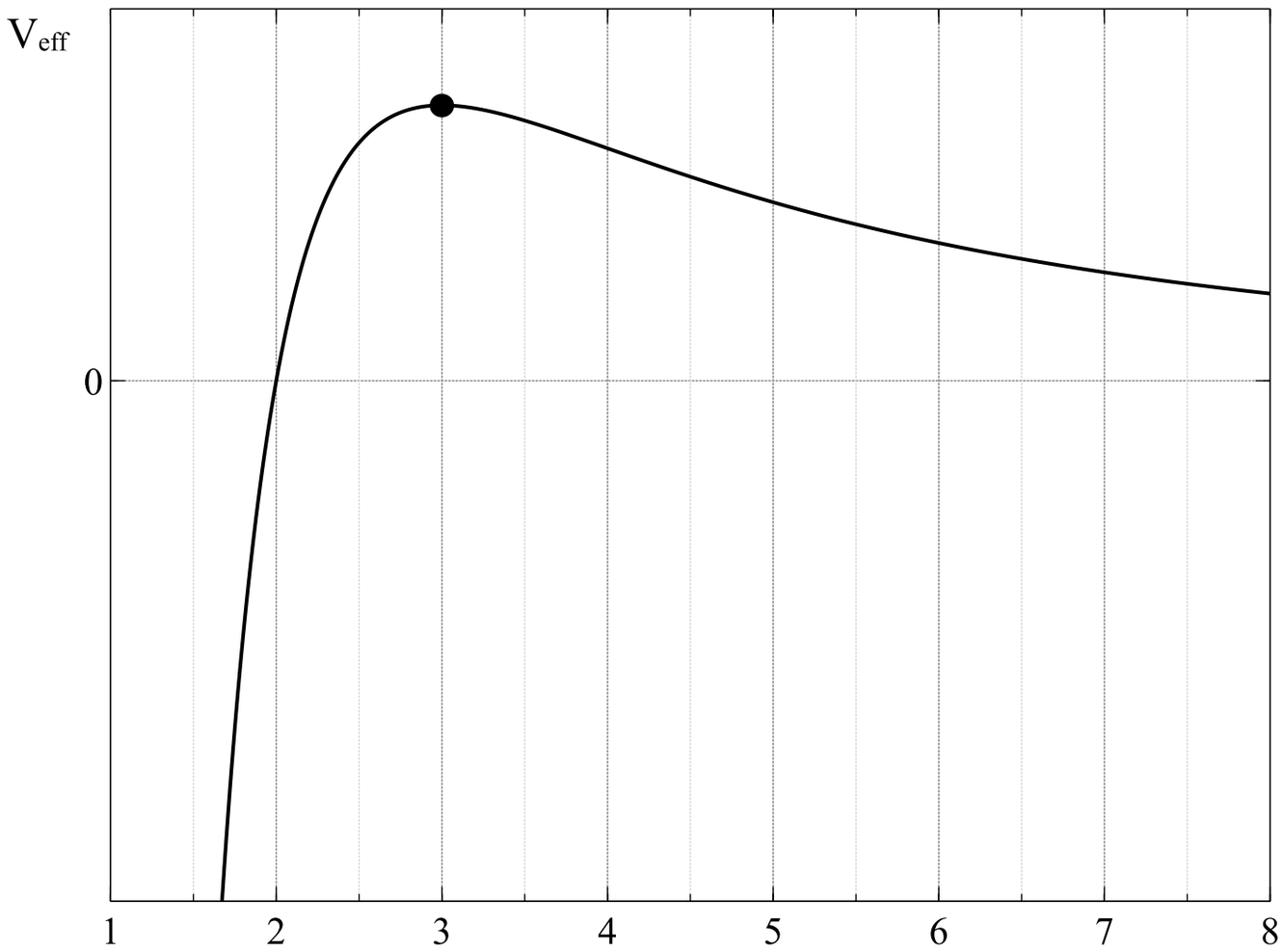
Circular Motion of Photons

$$\frac{d^2u}{d\varphi^2} + u = \frac{3GM}{c^2}u^2$$

circular orbit at $r = \frac{3GM}{c^2}$

Determine stability from energy equation

$$\frac{\dot{r}^2}{h^2} + \underbrace{\frac{1}{r^2}\left(1 - \frac{2\mu}{r}\right)}_{V_{\text{eff}}} = \frac{1}{b^2}$$



Maximum in potential

- Unstable orbit

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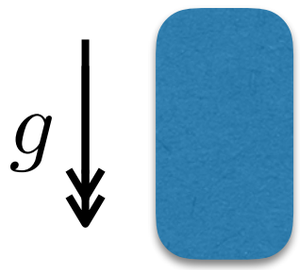
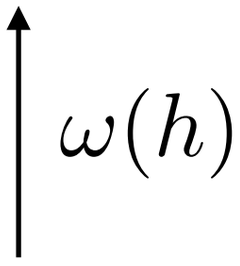
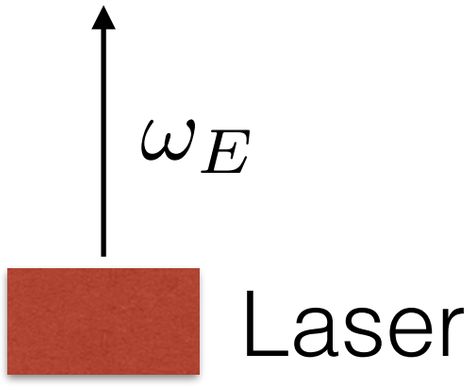
The Equivalence Principle (revisited)







At what frequency is the photon detected?

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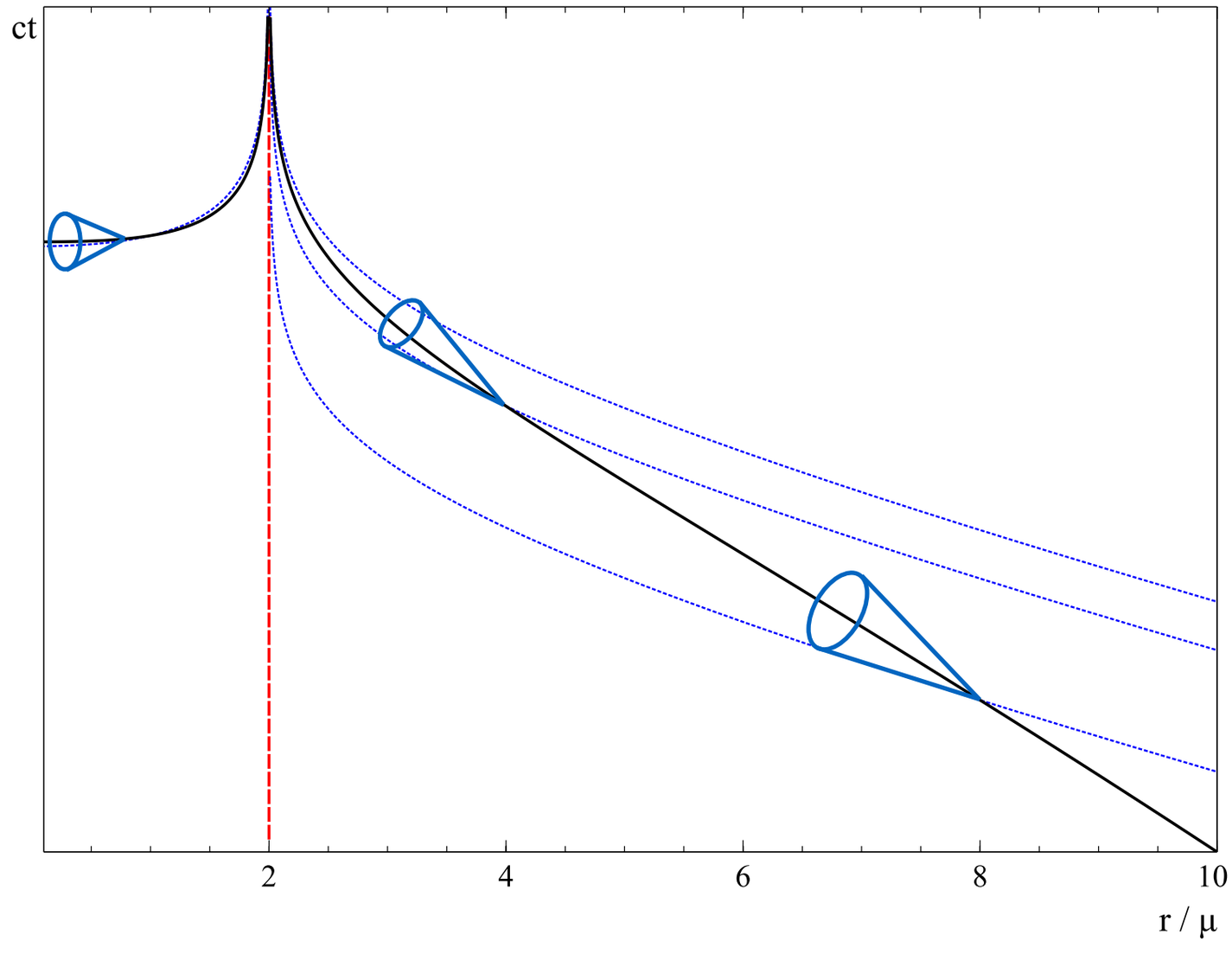
The Event Horizon

$$ds^2 = c^2\left(1 - \frac{2\mu}{r}\right)dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi\right)$$



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Rotating Black Holes

The Kerr Metric

$$ds^2 = c^2\left(1 - \frac{2\mu r}{\rho^2}\right)dt^2 + \frac{4\mu a c r \sin^2\theta}{\rho^2}dtd\varphi - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{2\mu a^2 r \sin^2\theta}{\rho^2}\right)\sin^2\theta d\varphi^2$$

$$\mu = \frac{GM}{c^2}$$

$$\rho^2 = r^2 + a^2\cos^2\theta$$

$$\Delta = r^2 - 2\mu r + a^2$$

Kerr 1964

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Rotating Black Holes

The Kerr Metric

$$ds^2 = \frac{\rho^2 \Delta}{\Sigma^2} c^2 dt^2 - \frac{\Sigma^2 \sin^2 \theta}{\rho^2} (d\varphi - \omega dt^2)^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2$$

$$\mu = \frac{GM}{c^2}$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta$$

$$\Delta = r^2 - 2\mu r + a^2$$

$$\Sigma^2 = \left(r^2 + a^2\right)^2 - a^2 \Delta \sin^2 \theta$$

$$\omega = \frac{2ar}{\sum^2}$$

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Rotating Black Holes

The Kerr Metric

$$ds^2 = \frac{\rho^2 \Delta}{\Sigma^2} c^2 dt^2 - \frac{\Sigma^2 \sin^2 \theta}{\rho^2} (d\varphi - \omega dt^2)^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2$$

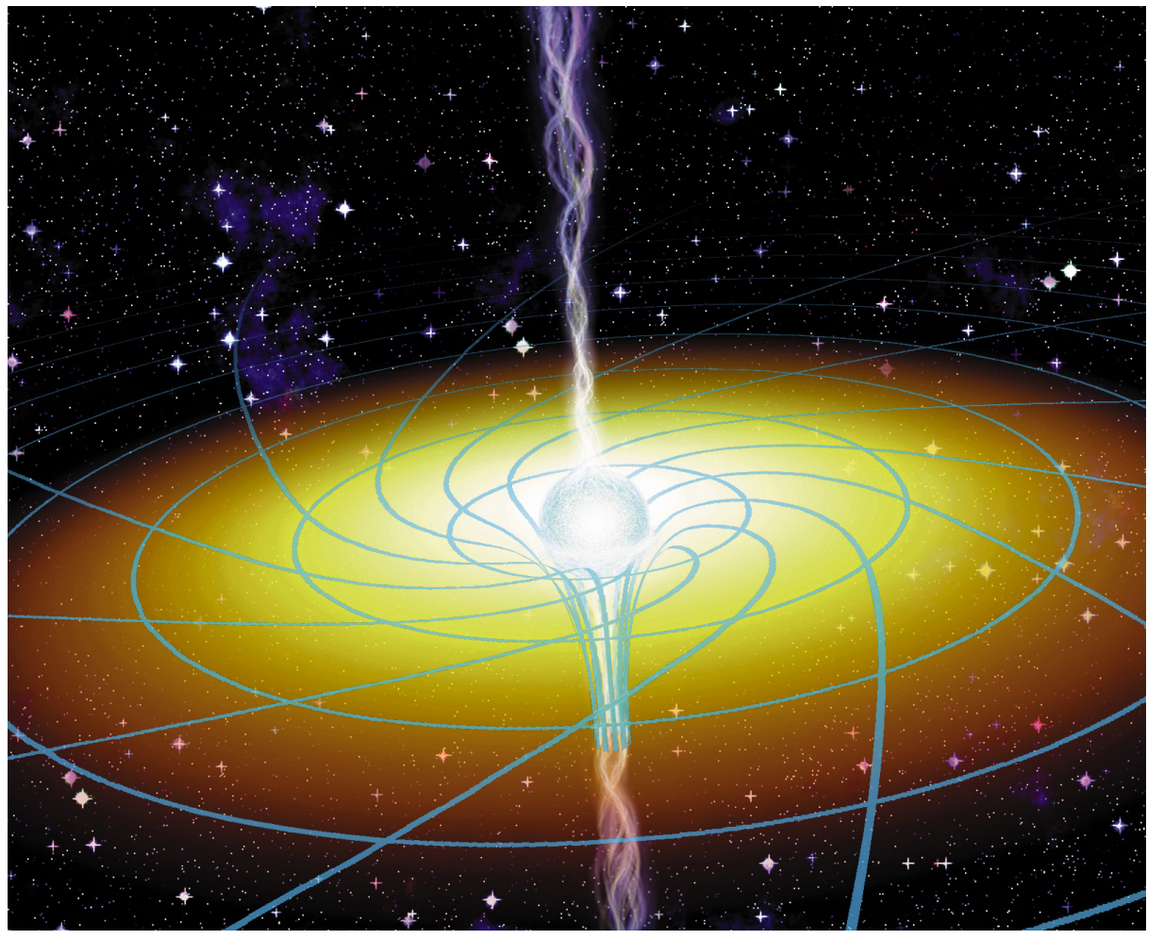
$$\mu = \frac{GM}{c^2}$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta$$

$$\Delta = r^2 - 2\mu r + a^2$$

$$\Sigma^2 = \left(r^2 + a^2\right)^2 - a^2 \Delta \sin^2 \theta$$

$$\omega = \frac{2ar}{\Sigma^2}$$



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Rotating Black Holes

The Kerr Metric

$$ds^2 = \frac{\rho^2 \Delta}{\Sigma^2} c^2 dt^2 - \frac{\Sigma^2 \sin^2 \theta}{\rho^2} (d\varphi - \omega dt^2)^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2$$

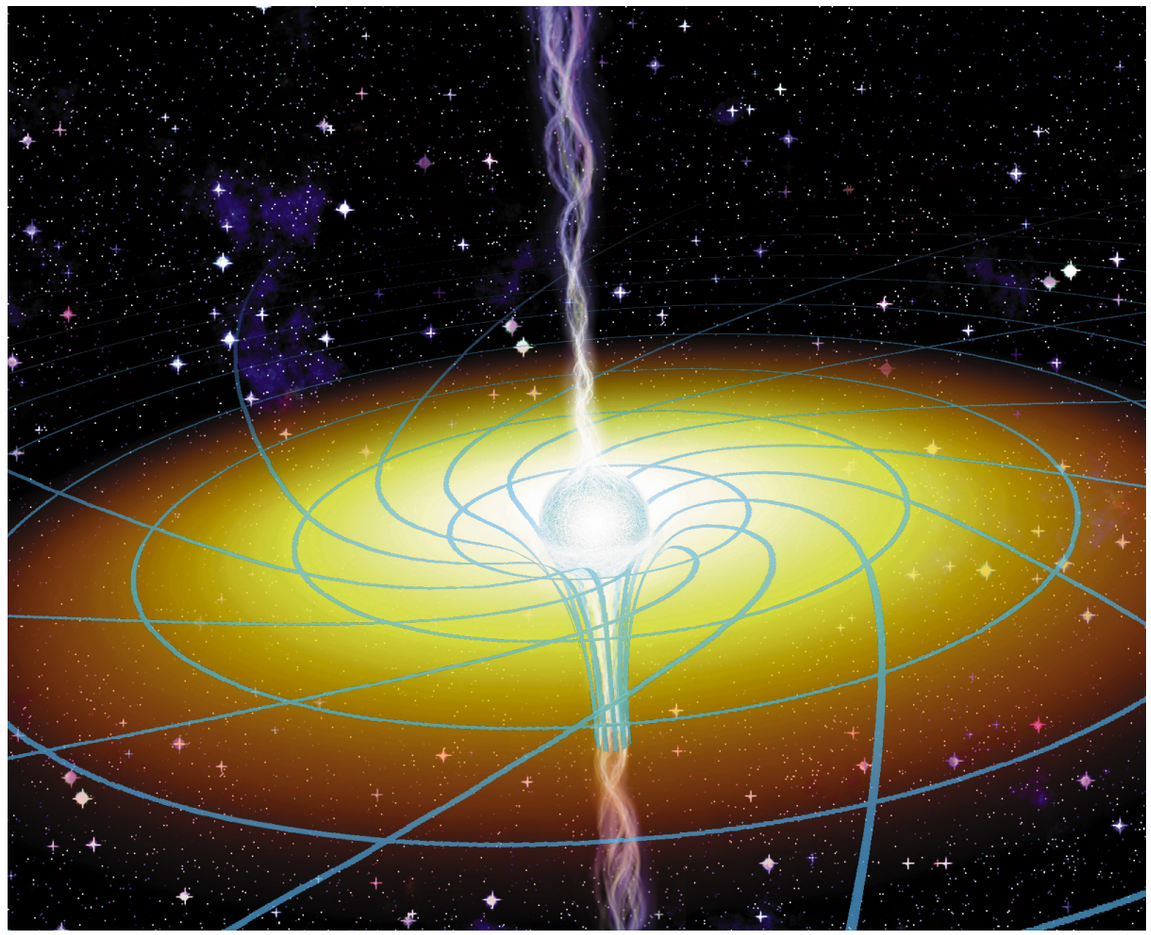
$$\mu = \frac{GM}{c^2}$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta$$

$$\Delta = r^2 - 2\mu r + a^2$$

$$\Sigma^2 = \left(r^2 + a^2\right)^2 - a^2 \Delta \sin^2 \theta$$

$$\omega = \frac{2ar}{\Sigma^2}$$



$$J = Mac$$

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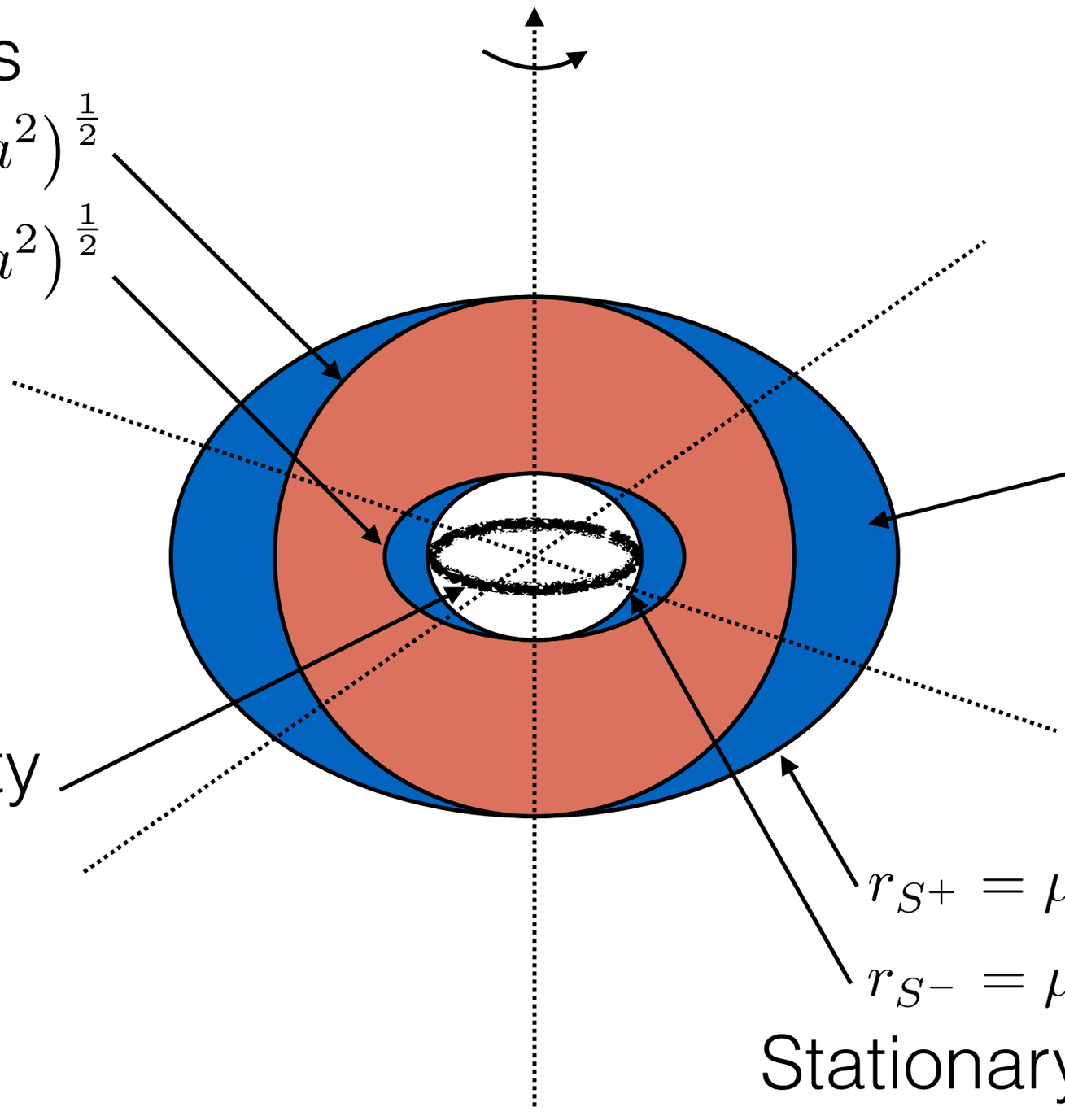
Structure of a Kerr Black Hole

Event horizons

$$r_+ = \mu + \left(\mu^2 - a^2\right)^{\frac{1}{2}}$$

$$r_- = \mu - \left(\mu^2 - a^2\right)^{\frac{1}{2}}$$



Ergosphere

Ring singularity

$$r_{S+} = \mu + \left(\mu^2 - a^2 \cos^2\theta\right)$$

$$r_{S-} = \mu - \left(\mu^2 - a^2 \cos^2\theta\right)$$

Stationary limit surfaces

(Infinite redshift)

Kerr 1964

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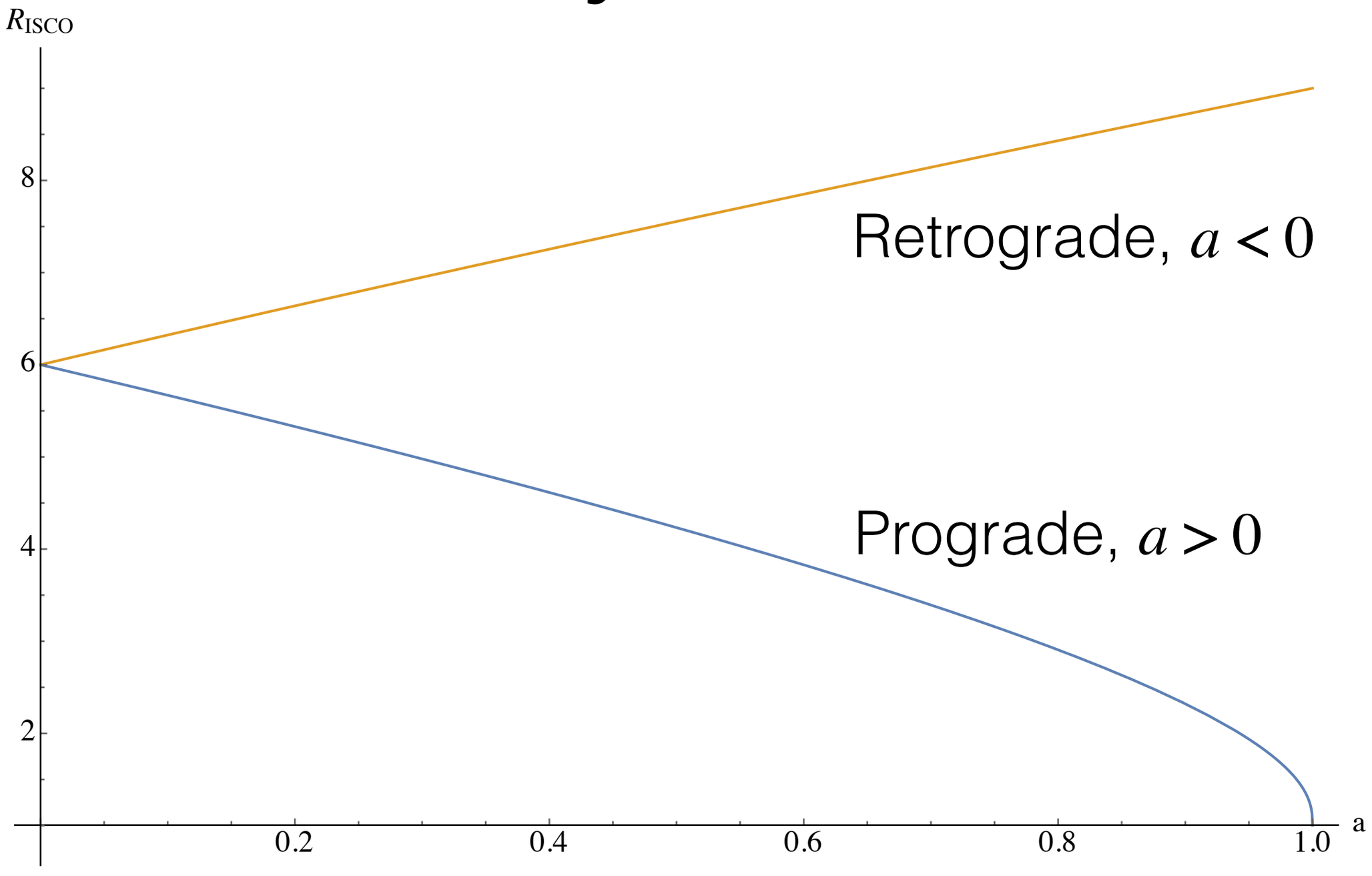
Stability of Orbits



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What would you see as you fall into a black hole?

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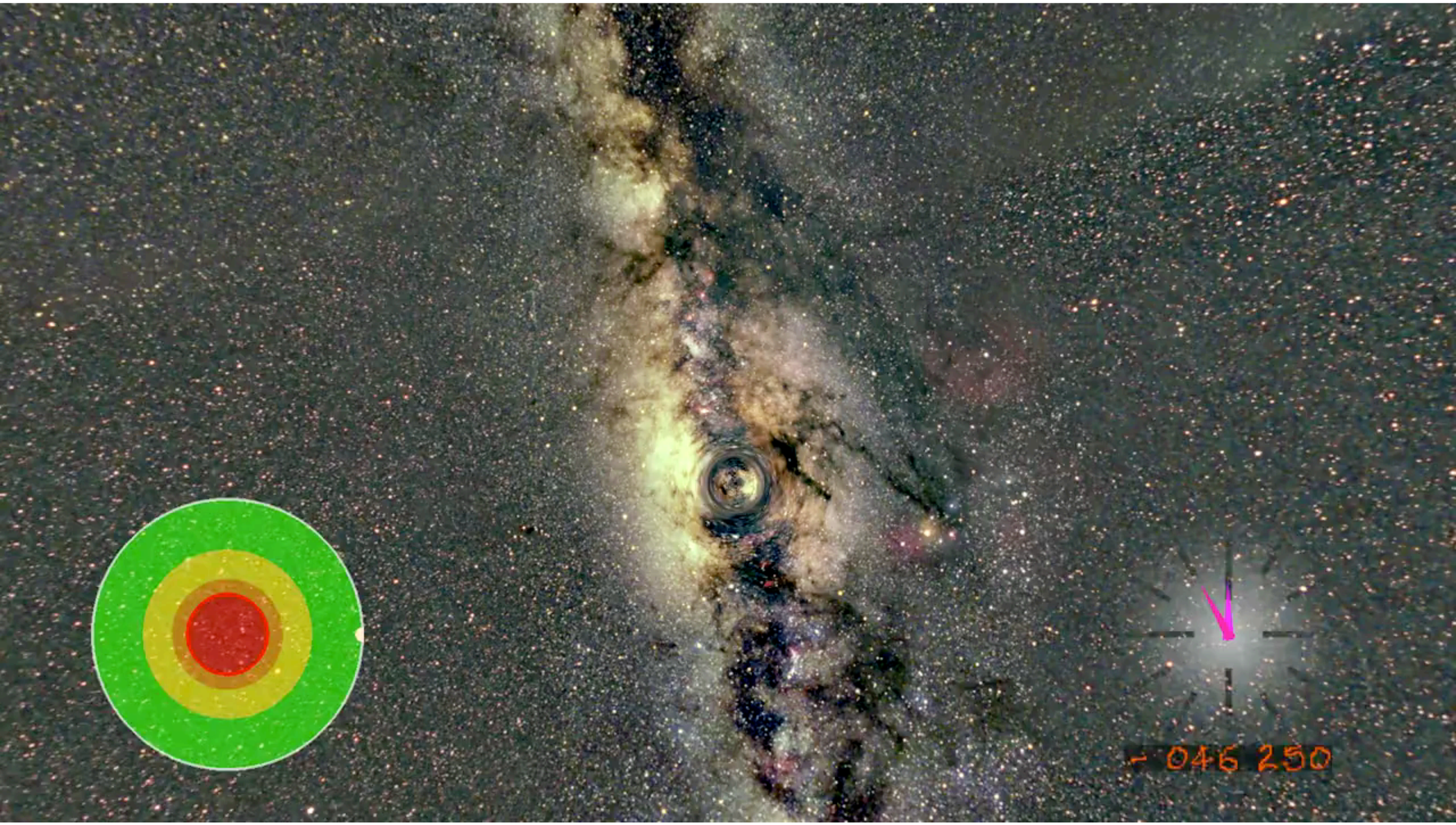

Andrew Hamilton's Inside Black Holes - http://jila.colorado.edu/~ajsh/insidebh

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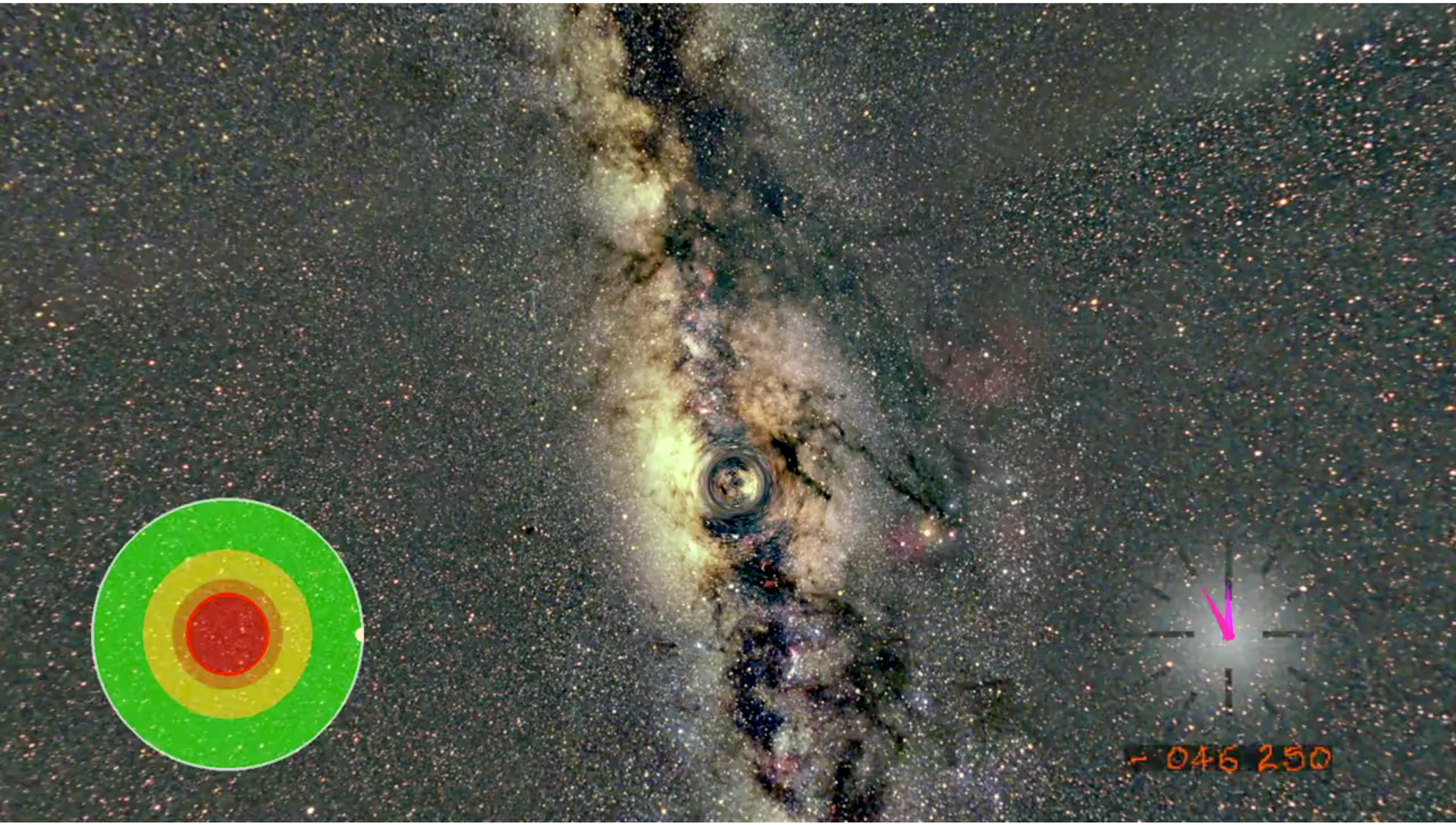

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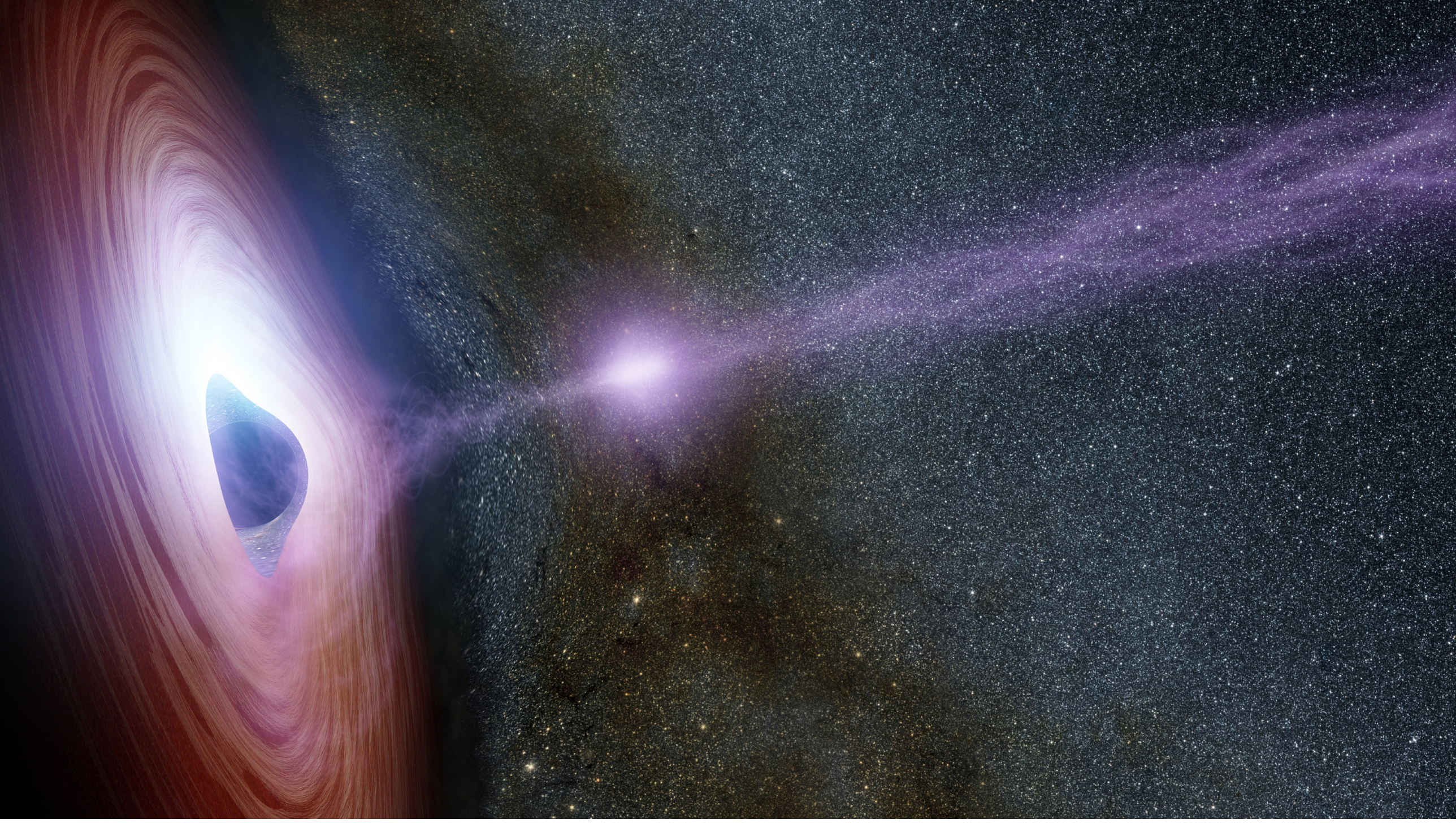

Accretion

Powering some of the brightest objects in the Universe

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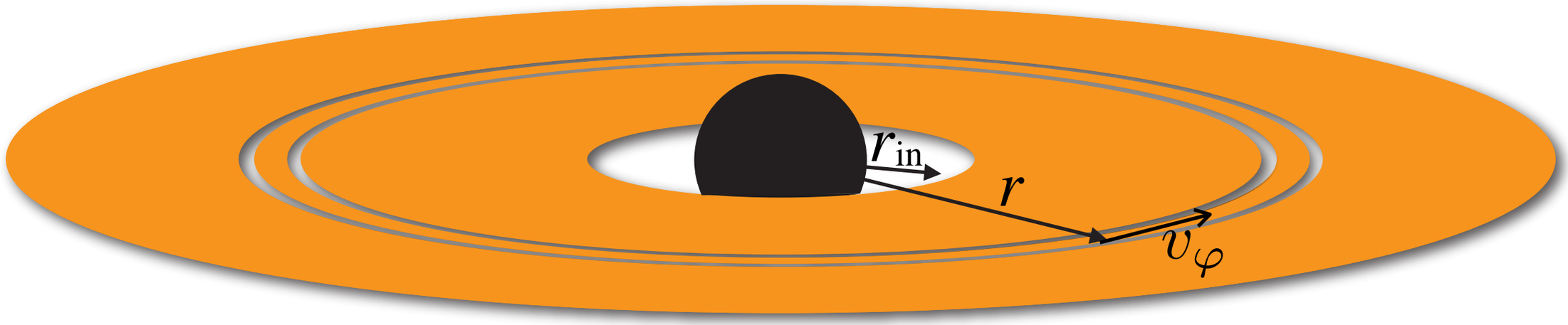
Accretion Discs



To transfer matter inwards, need to transfer angular momentum outwards

$$v_\varphi = r\Omega = \sqrt{\frac{GM}{r}} \qquad\qquad j = r^2\Omega = \sqrt{GMr}$$

Total angular momentum transfer rate $\dot{J} = \dot{M}\sqrt{GMr}$

If some fraction β can be accreted from inner edge

$$\dot{J} = \dot{M}\left(\sqrt{GMr} - \beta\sqrt{GMr_{\rm in}}\right)$$

Shakura & Sunyaev 1974, Pringle 1981

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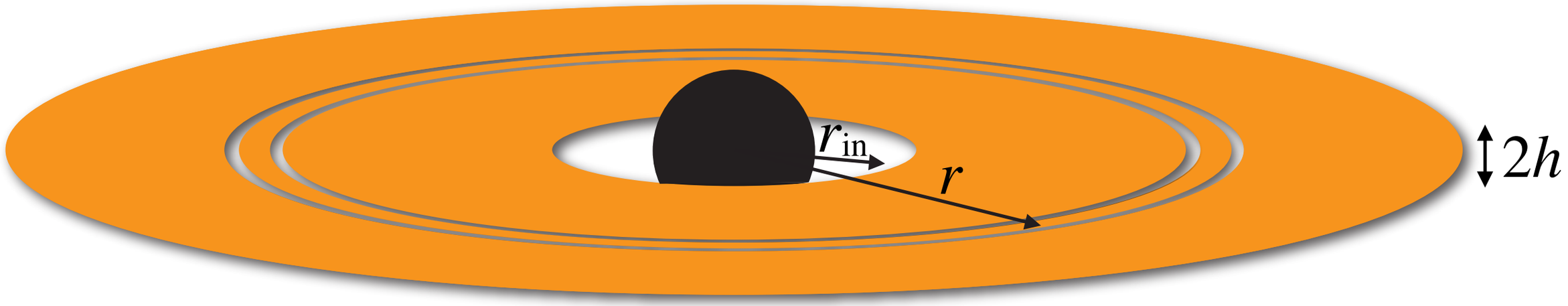
Accretion Discs



Angular momentum in transferred by viscous stresses

$$s_\varphi = \eta \frac{dv_\varphi}{dr}$$

Rate of change of angular momentum equal to torque

$$s_\varphi (2\pi r 2h) r = \dot{M}\left(\sqrt{GMr} - \beta\sqrt{GMr_{\rm in}}\right)$$

$$s_\varphi = \frac{\dot{M}(\sqrt{GMr} - \beta\sqrt{GMr_{\rm in}})}{4\pi r^2 h}$$

Shakura & Sunyaev 1974, Pringle 1981

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Disc Emission

$$\text{Work Done} = \text{Force} \times \text{Velocity}$$

$$\frac{\text{Power}}{\text{Unit Volume}} = \frac{\text{Force} \times \text{Velocity}}{\text{Volume}} = \frac{\text{Force}}{\text{Area}} \times \frac{\text{Velocity}}{\text{Radius}}$$

$$\dot{Q} = s_\varphi \frac{dv_\varphi}{dr} = s_\varphi r \frac{d\Omega}{dr} = \frac{3}{2}\Omega s_\varphi \qquad \frac{dv_\varphi}{dr} = \frac{d(\Omega r)}{dr} = \Omega + r\frac{d\Omega}{dr}$$

So the power output per unit volume of the disc

$$\dot{Q} = \frac{3}{2}\Omega \frac{\dot{M}(\sqrt{GMr} - \beta\sqrt{GMr_{\text{in}}})}{4\pi r^2 h}$$

Shakura & Sunyaev 1974, Pringle 1981

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Disc Emission

- From the power output per unit volume, can integrate through the height of the disc to get the flux (power per unit area)
- Half comes out the top of the disc and half out the bottom

$$F = \frac{1}{2} 2h\dot{Q}$$

$$F = \frac{3GM\dot{M}}{8\pi r^3}\left(1 - \beta\sqrt{\frac{r_{\rm in}}{r}}\right)$$

Shakura & Sunyaev 1974, Pringle 1981

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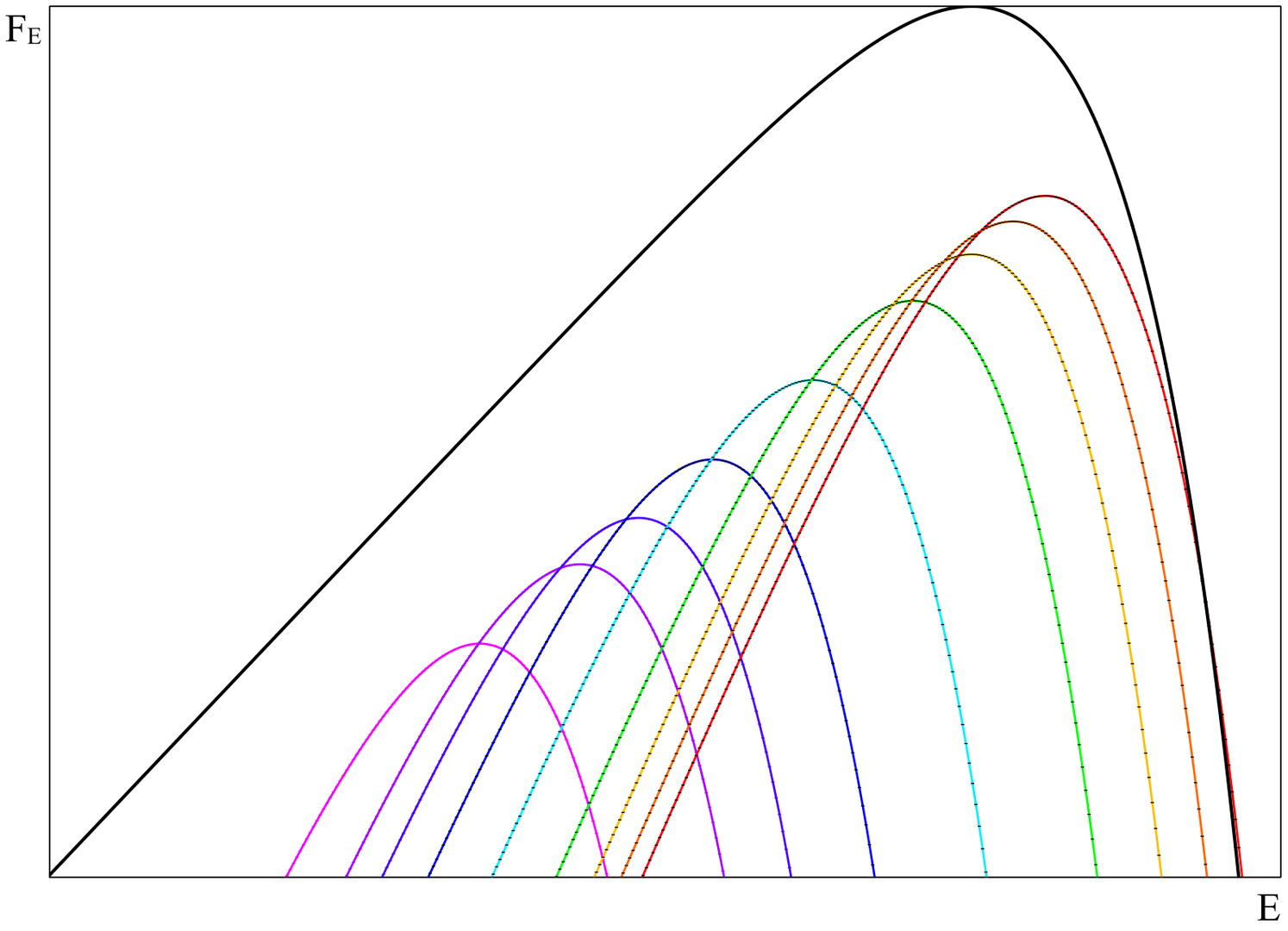
The Disc Spectrum

Assume disc is optically thick so each part of the disc radiates as a black body



$$F = \sigma T^4$$

$$T(r) = \left[\frac{F(r)}{\sigma}\right]^{\frac{1}{4}}$$

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The Viscosity

- Viscosity expressed in terms of the hydrostatic pressure in the disc $s_\varphi = \alpha p$ with $\alpha < 1$

- $\alpha \sim 0.1$ to produce accretion rates high enough to produce observed luminosity

- Cannot be explained by viscous friction…

Shakura & Sunyaev 1974

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Accretion Efficiency

$$L = \epsilon \dot{M} c^2$$

Assume the gravitational potential lost between infinity and the innermost stable orbit is radiated away (all radiation from within is lost into black hole)

Newtonian orbits:

$$L = \frac{GM\dot{M}}{r} - \frac{1}{2}\dot{M}v^2 = \frac{GM\dot{M}}{2r}$$

$$\epsilon = \frac{L}{\dot{M}c^2} \qquad r = 6\mu \qquad \epsilon = \frac{1}{12}$$

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Accretion Efficiency

$$L = \epsilon \dot{M} c^2$$

Assume the gravitational potential lost between infinity and the innermost stable orbit is radiated away (all radiation from within is lost into black hole)

Schwarzschild spacetime:

$$\epsilon = \frac{E - m_0 c^2}{m_0 c^2} \qquad k = \frac{E}{m_0 c^2} \qquad \epsilon = 1 - k$$

$$k = \frac{(1 - 2\mu/r)}{(1 - 3\mu/r)^{\frac{1}{2}}}$$

ϵ = 5.7% for a Schwarzschild black hole

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How does the efficiency of a rotating black hole compare?

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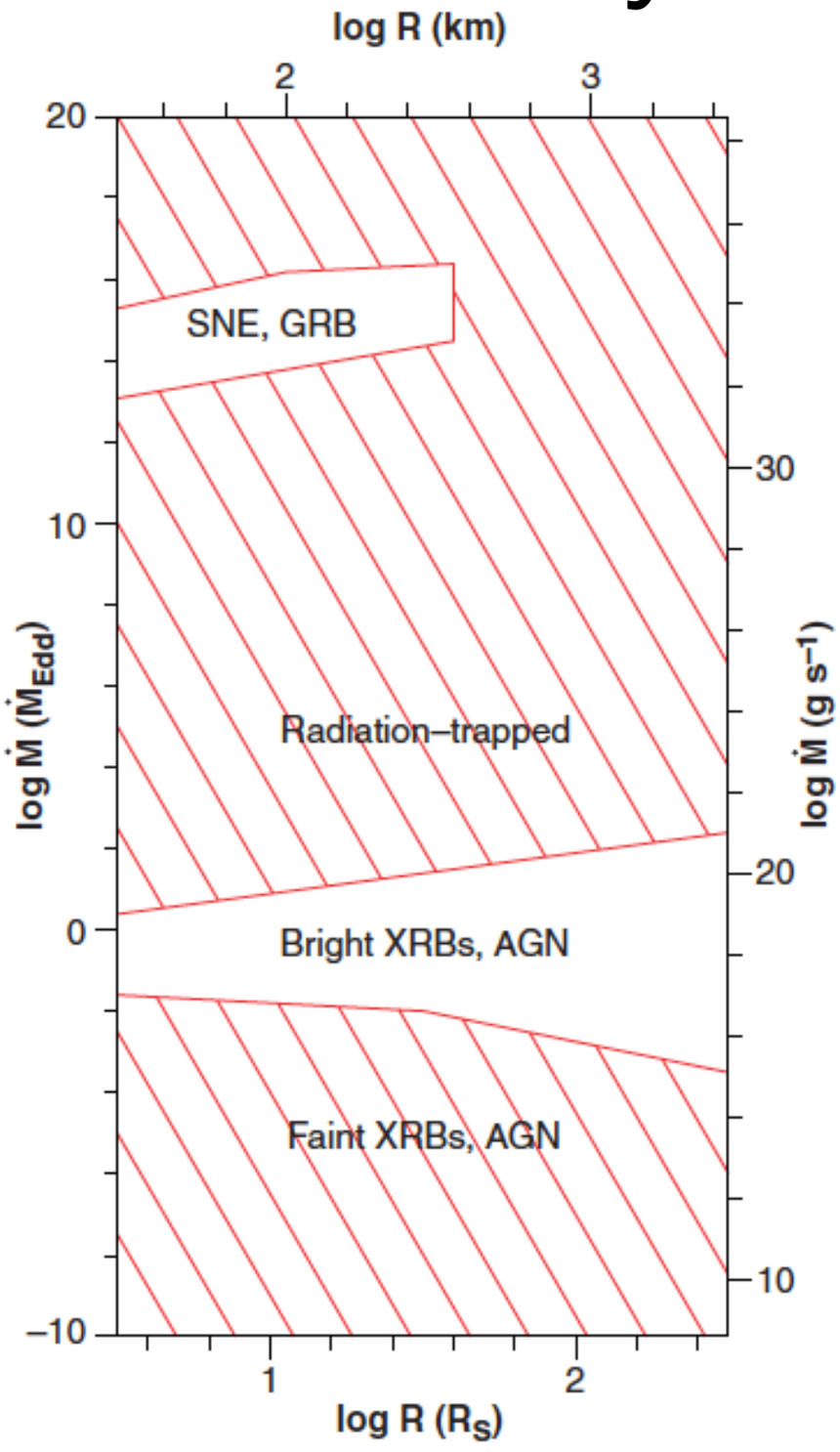
Radiatively Inefficient Accretion



Narayan 1998, Narayan & Quataert 2005

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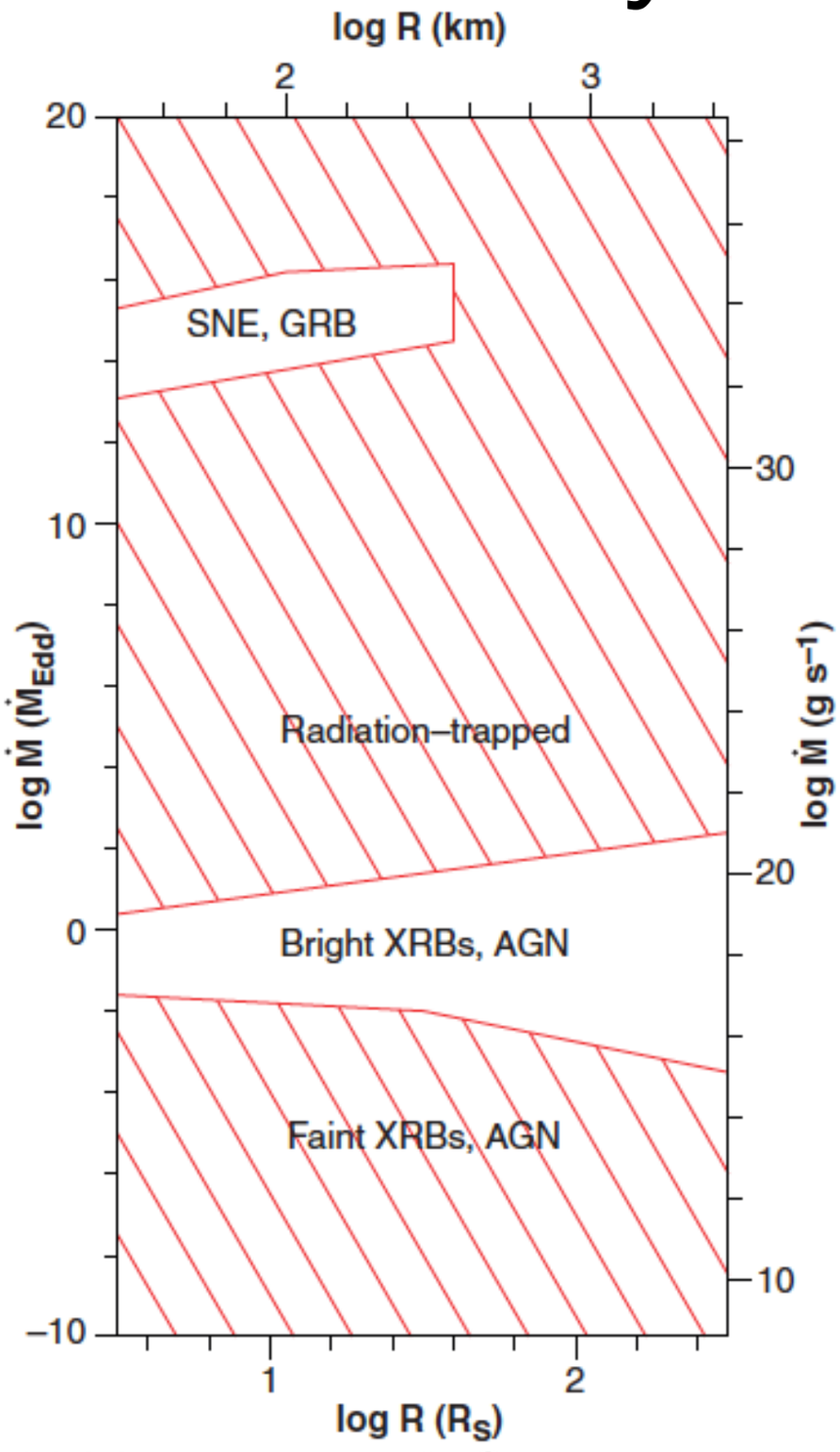
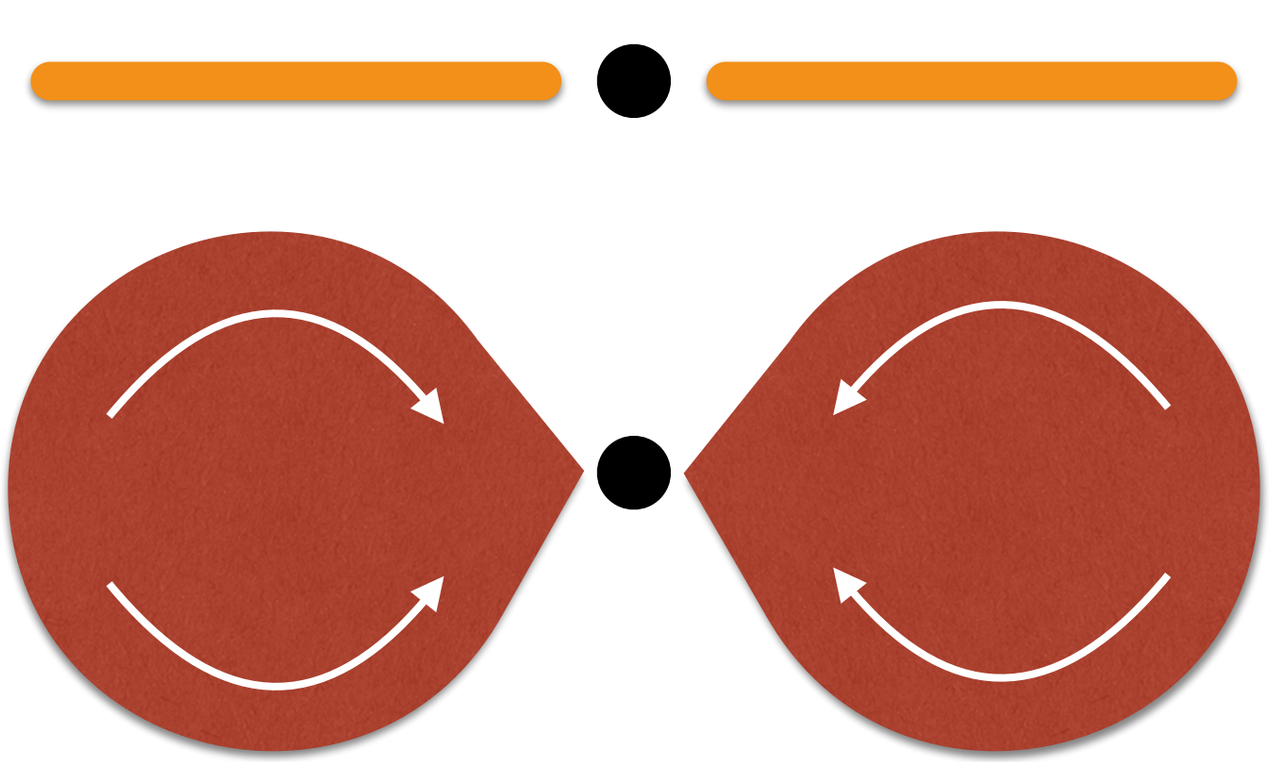
Radiatively Inefficient Accretion





Narayan 1998, Narayan & Quataert 2005

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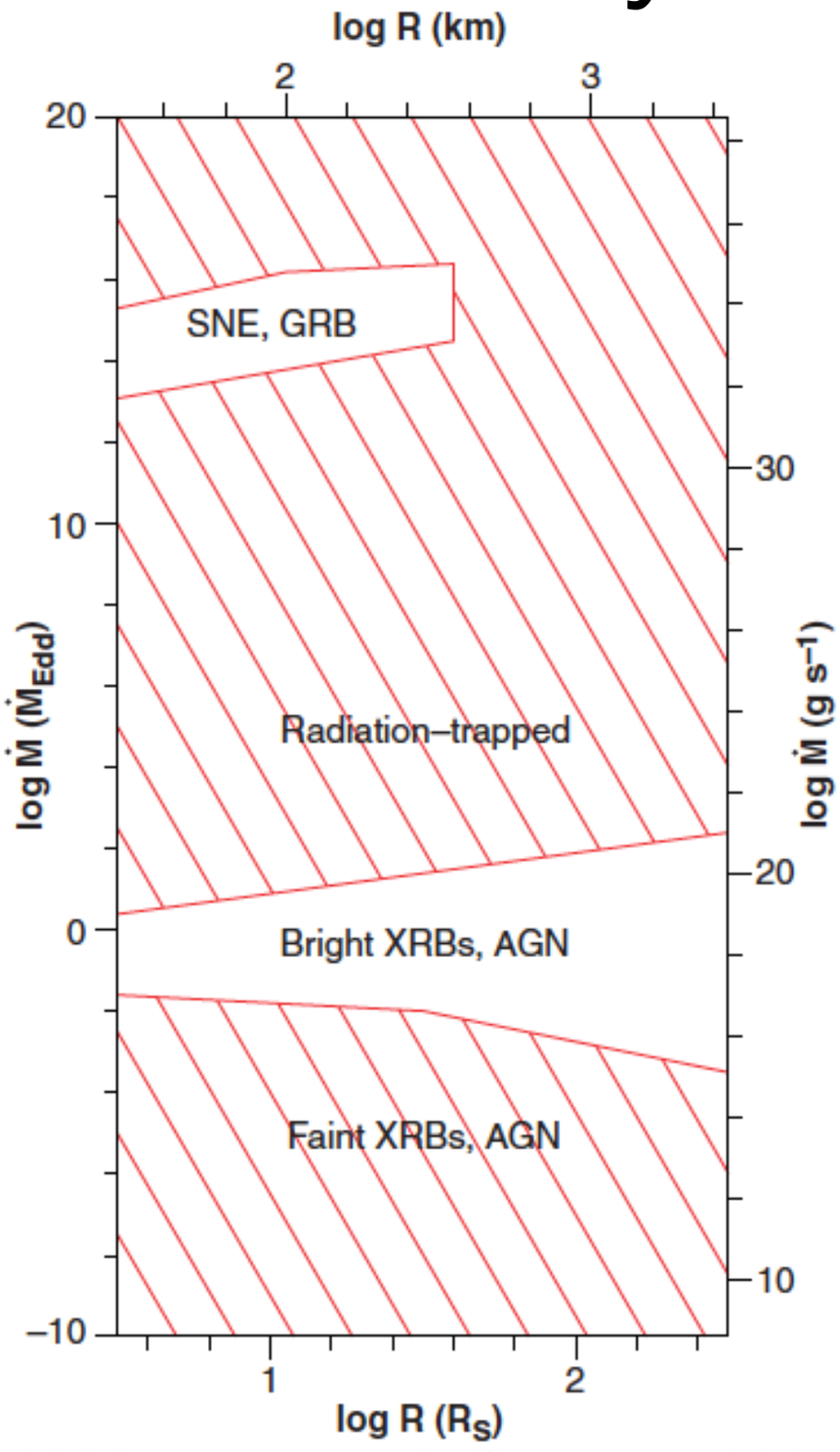
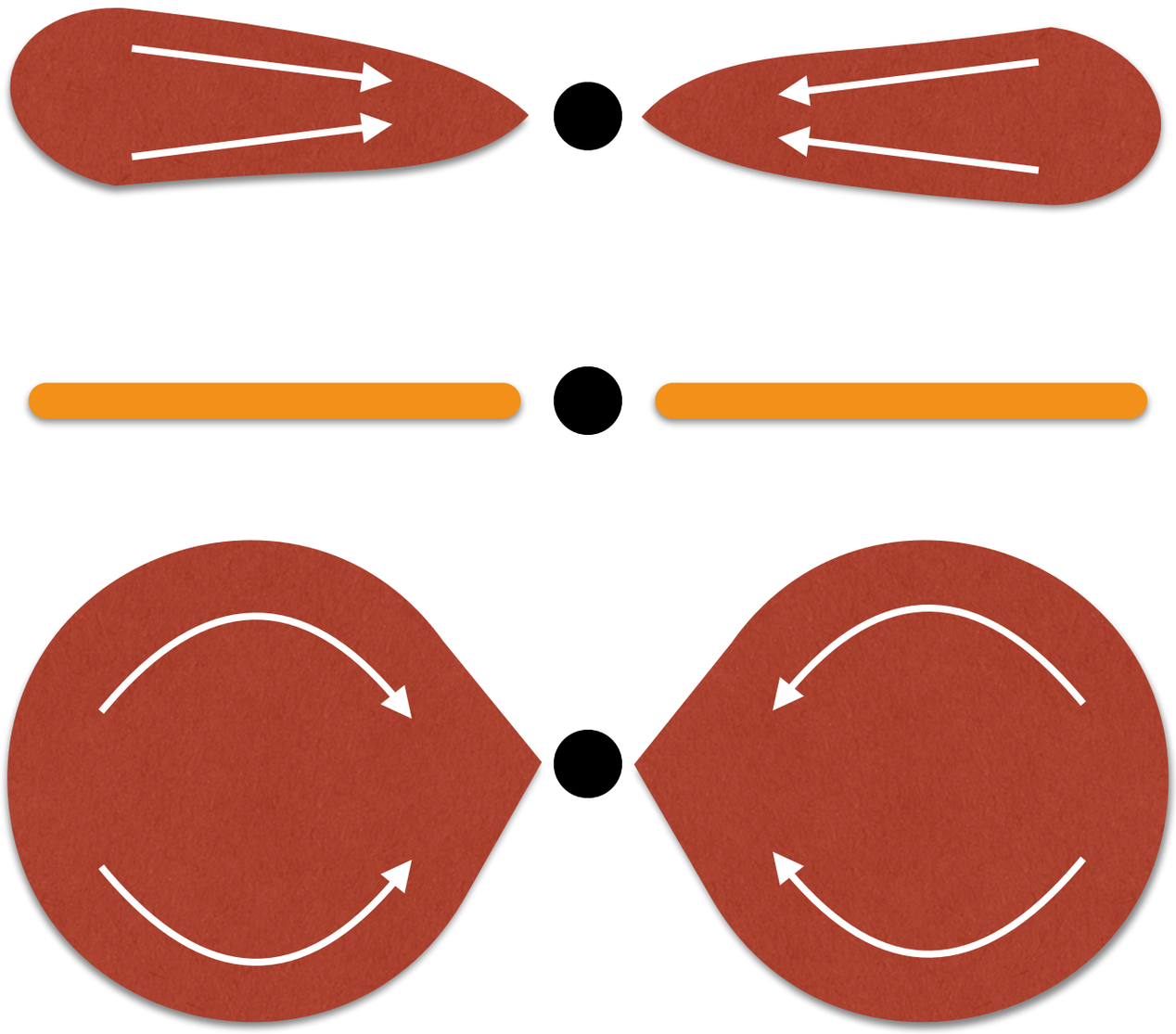
Radiatively Inefficient Accretion





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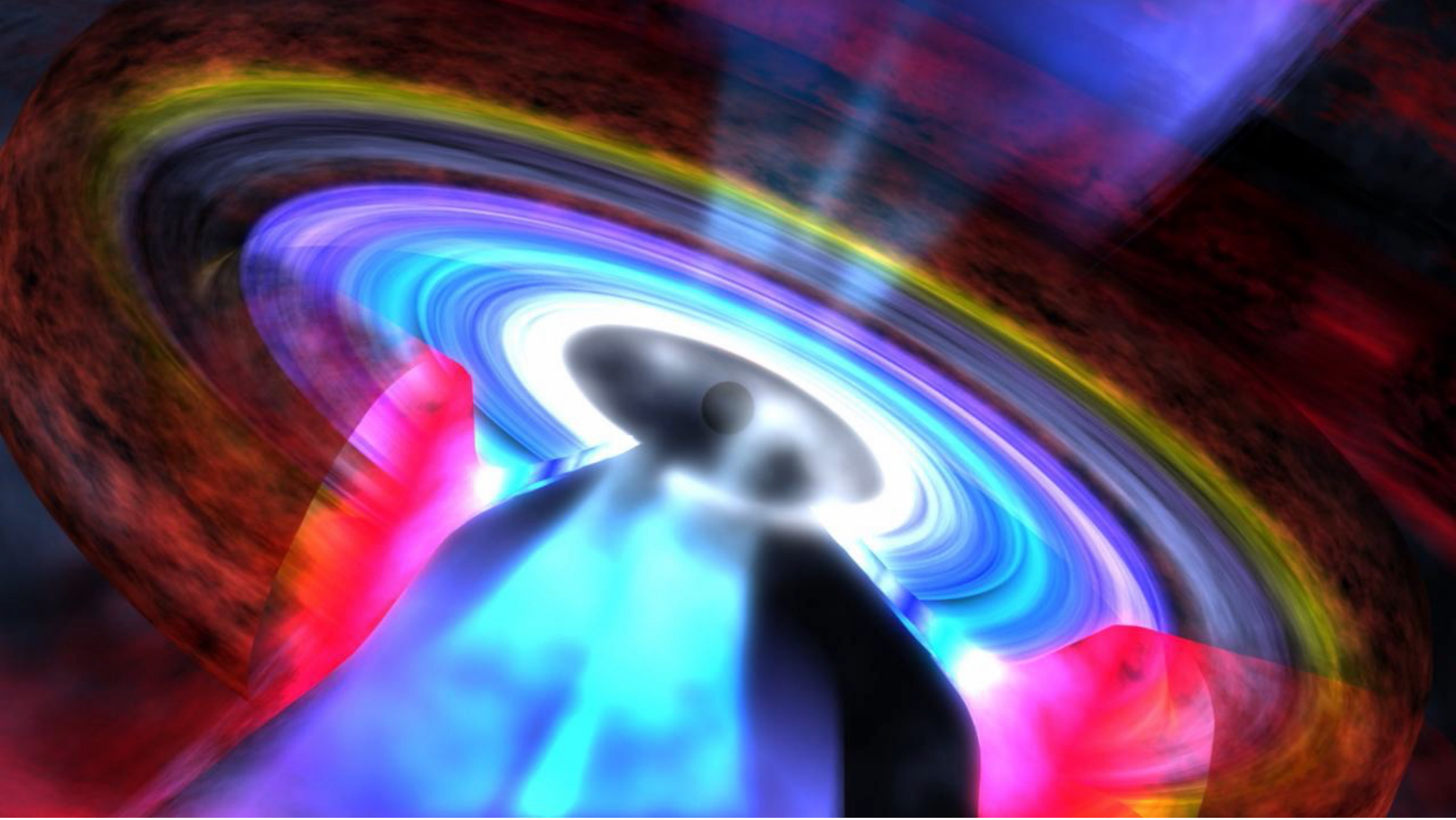

Building a more realistic picture of the accretion disc

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Magnetohydrodynamics

If the accretion flow is made of an ionised plasma, it can be described by magnetohydrodynamics (MHD)

- Continuity equation (mass conservation)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$

- Charge conservation

$$\frac{\partial q}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

- Momentum equation

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} \right) = q\vec{E} + \vec{j} \wedge \vec{B} - \vec{\nabla} p$$

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Magnetohydrodynamics

Electric and magnetic fields obey Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = \frac{q}{\epsilon_0}$$

$$\vec{\nabla} \wedge \vec{E} = -\frac{\partial B}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \wedge \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

And Ohm's law $\vec{j} = \sigma(\vec{E} + \vec{u} \wedge \vec{B})$

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Ideal Magnetohydrodynamics

- Charge neutrality – charge imbalance causes motion of electrons to balance it out. Charge imbalance oscillates at the plasma frequency - effective shielding length
- Non-relativistic flow $\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \ll \vec{\nabla} \wedge \vec{B}$ $\vec{\nabla} \wedge \vec{B} = \mu_0 \vec{j}$
- Infinite conductivity – can neglect diffusion of the magnetic field $\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \wedge (\vec{u} \wedge \vec{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \vec{B}$

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Ideal Magnetohydrodynamic Equations

- Taking the fluid equations and Maxwell's equations with the assumptions of ideal magnetohydrodynamics and eliminate E

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} \right) = \vec{j} \wedge \vec{B} - \vec{\nabla} p$$

$$\vec{\nabla} \wedge (\vec{u} \wedge \vec{B}) = \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \wedge \vec{B} = \mu_0 \vec{j}$$

$$p = K \rho^{\gamma} \qquad \gamma = \frac{5}{3}$$

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The Magnetic Field

$$\vec{F}_{\mathrm{mag}} = \vec{j} \wedge \vec{B}$$

$$\vec{\nabla} \wedge \vec{B} = \mu_0 \vec{j}$$

$$\vec{F}_{\mathrm{mag}} = \frac{1}{\mu_0}\left[-\vec{\nabla}\left(\frac{B^2}{2}\right) + (\vec{B}\cdot\vec{\nabla})\vec{B}\right]$$

- Magnetic pressure $p_{\mathrm{mag}} = \dfrac{B^2}{2\mu_0}$
- Magnetic tension from $(\vec{B}\cdot\vec{\nabla})\vec{B}$

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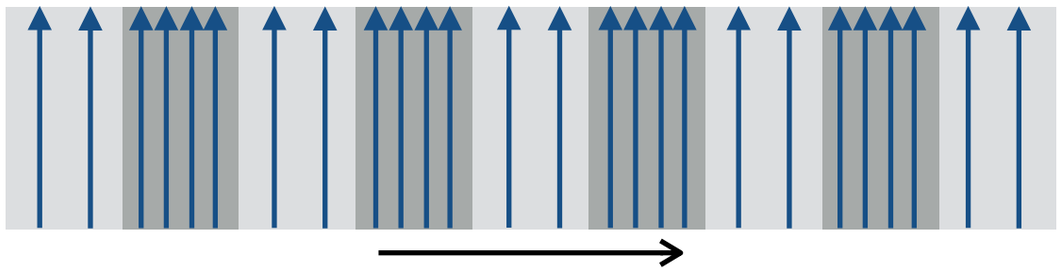
MHD Waves

Across field lines

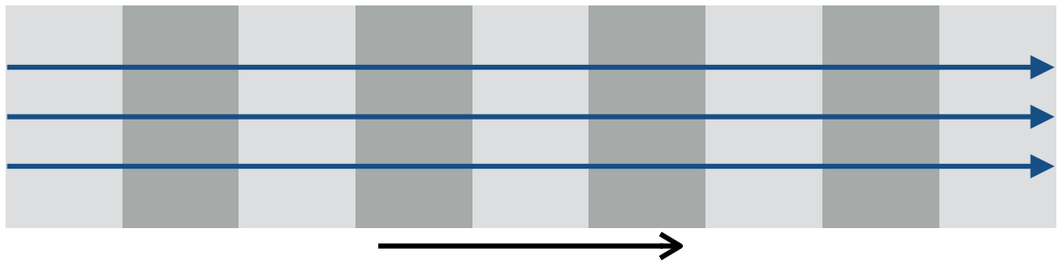
Longitudinal

$$v_p = \sqrt{c_s^2 + v_A^2}$$



Longitudinal

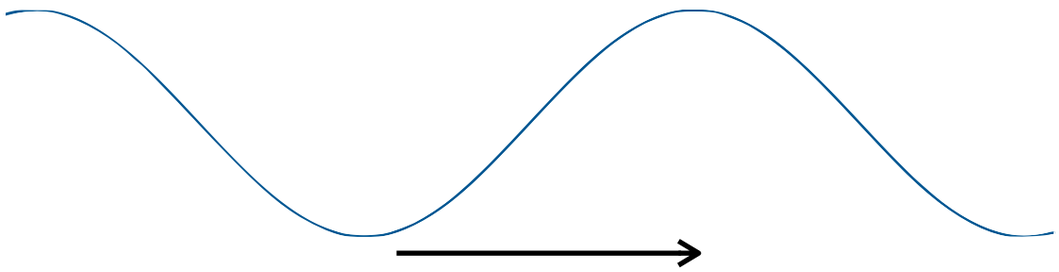
$$v_p = c_s$$



Along field lines

Transverse

$$v_p = v_A$$



$v_A = \sqrt{\dfrac{B^2}{\mu_0 \rho}}$ is the Alfvén velocity (equal kinetic and magnetic energy)

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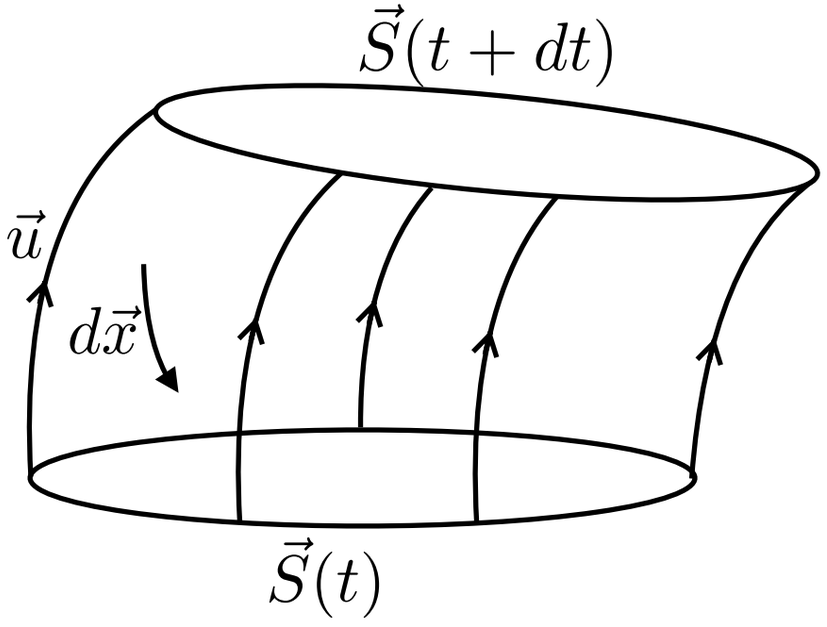
Magnetic Flux Freezing

Flux through a surface co-moving with the fluid (velocity $\vec{u}$)



$$\frac{d\Phi}{dt} = \lim_{dt\to 0}\left[\int_{\vec{S}(t+dt)} \vec{B}(t+dt)\cdot d\vec{S} - \int_{\vec{S}(t)} \vec{B}(t)\cdot d\vec{S}\right]$$

The flux through the total closed surface (ζ is the edge, dx a line within it) must be zero

$$\int_{\vec{S}(t+dt)} \vec{B}(t+dt)\cdot d\vec{S} + \int_{\zeta} \vec{B}(t+dt)\cdot d\vec{x}\wedge\vec{u}\,dt - \int_{\vec{S}(t)} \vec{B}(t+dt)\cdot d\vec{S} = 0$$

Use Stokes' theorem to integrate around the curve dx that encloses the surface S(t) and use MHD equations

$$\frac{d\Phi}{dt} = \int_{\vec{S}(t)}\left[\frac{\partial\vec{B}}{\partial t} - \vec{\nabla}\wedge(\vec{u}\wedge\vec{B})\right]\cdot d\vec{S} = \frac{d\Phi}{dt} = \int_{\vec{S}(t)}\left(\frac{1}{\mu_0\sigma}\nabla^2\vec{B}\right)\cdot d\vec{S}$$

If conductivity is infinite $\frac{d\Phi}{dt} = 0$, flux moves with fluid

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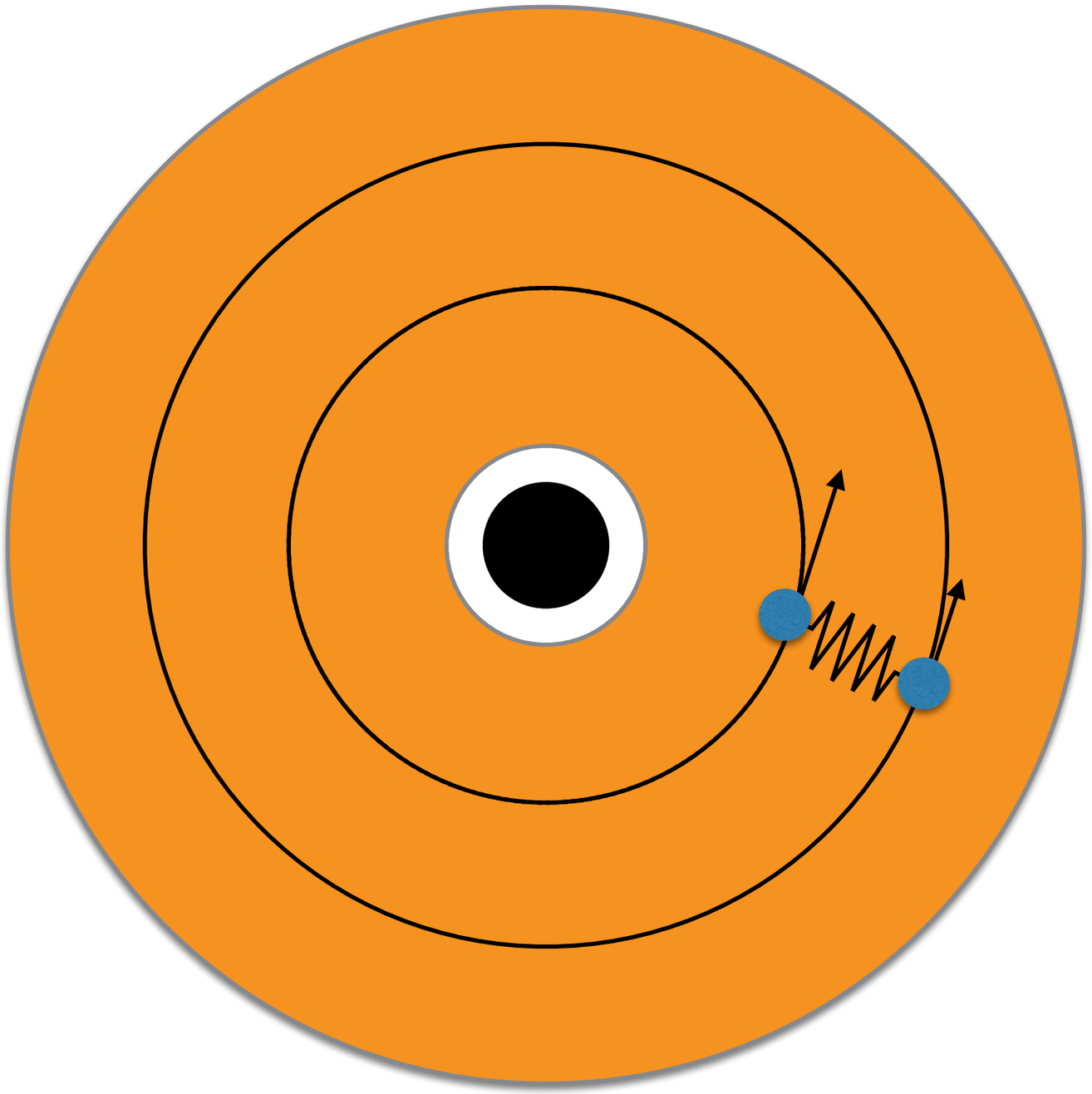
Magnetic Viscosity

Magnetorotational Instability (MRI)



Balbus & Hawley 1991

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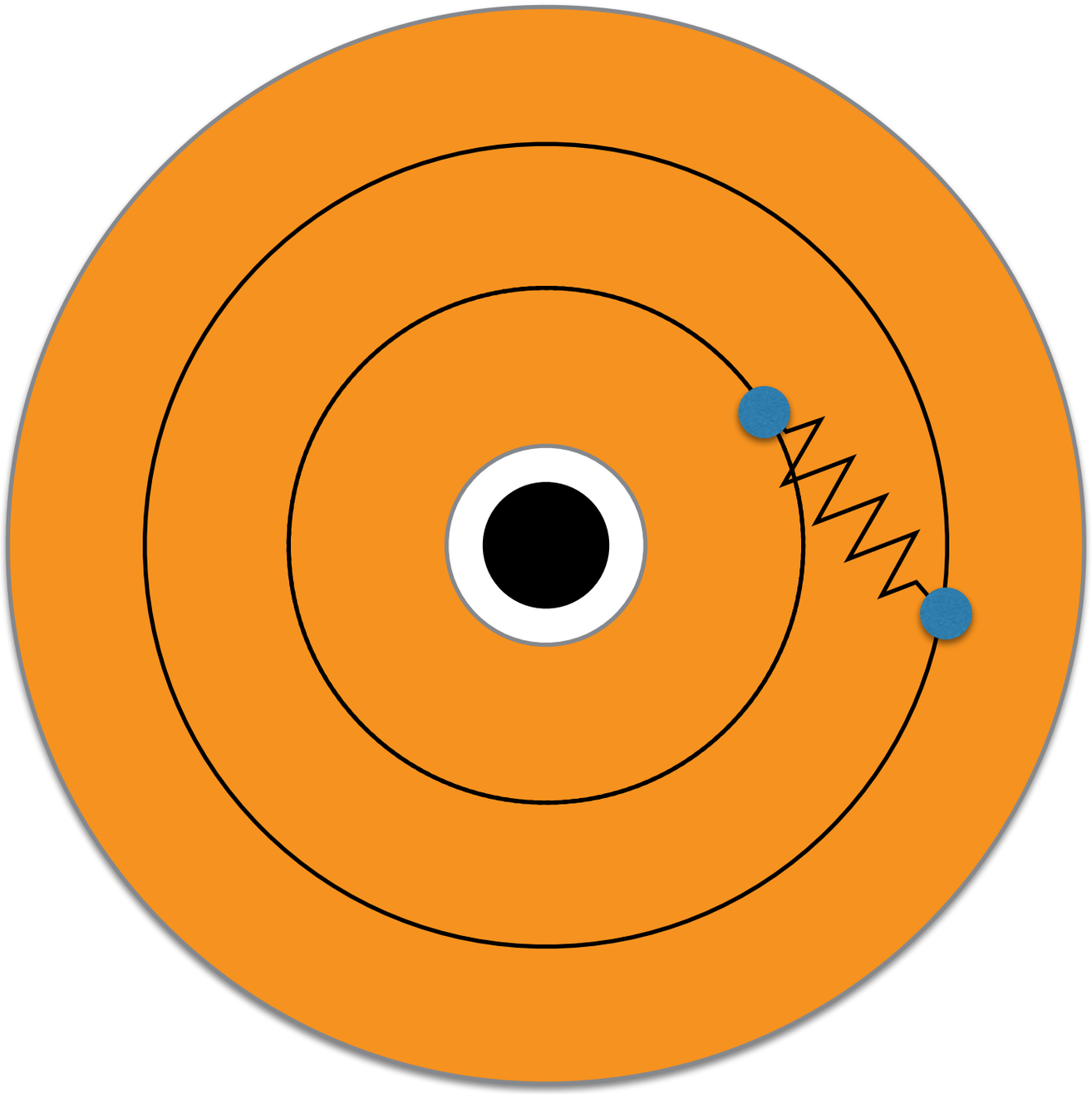
Magnetic Viscosity

Magnetorotational Instability (MRI)



Balbus & Hawley 1991

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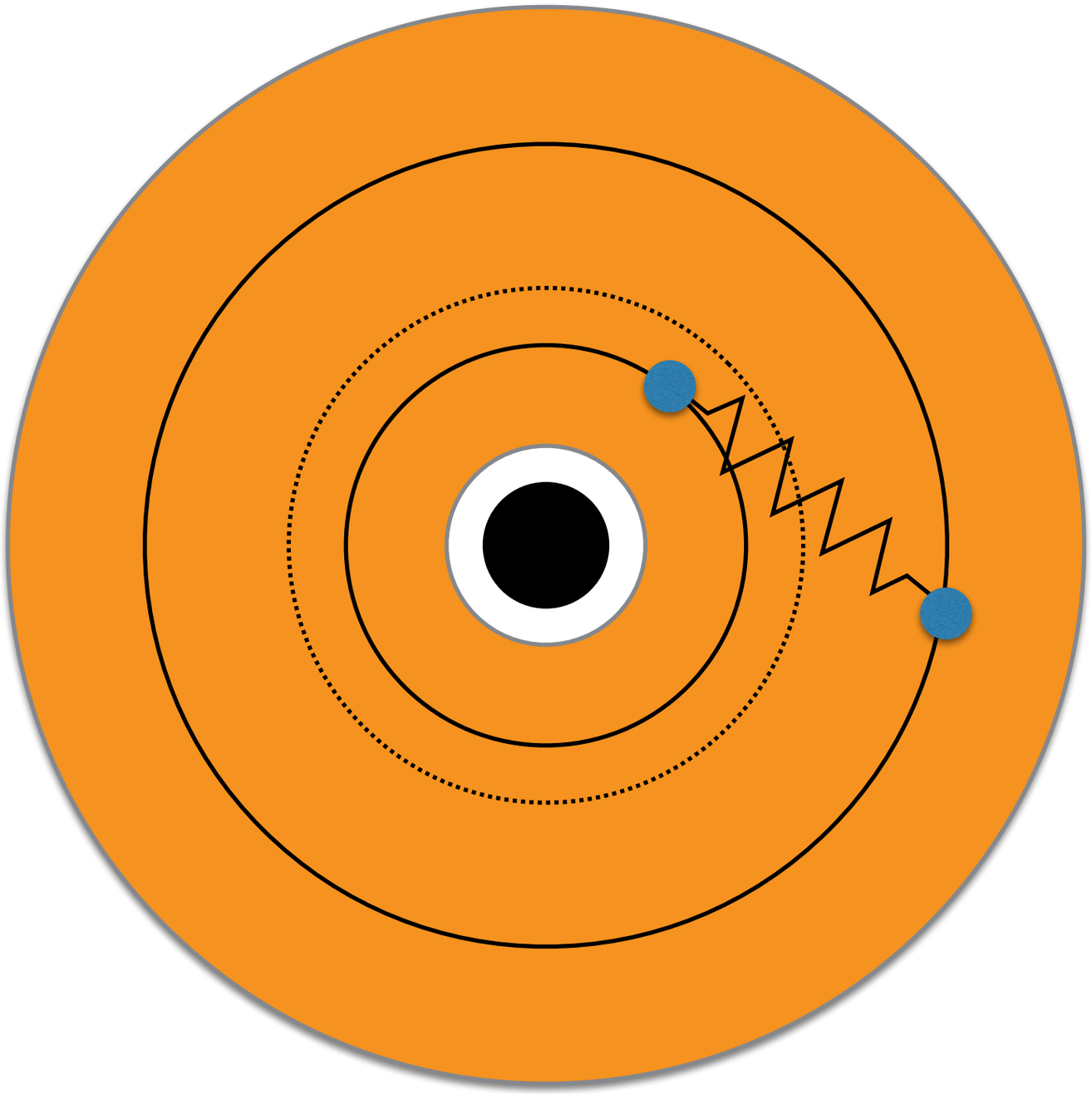
Magnetic Viscosity

Magnetorotational Instability (MRI)



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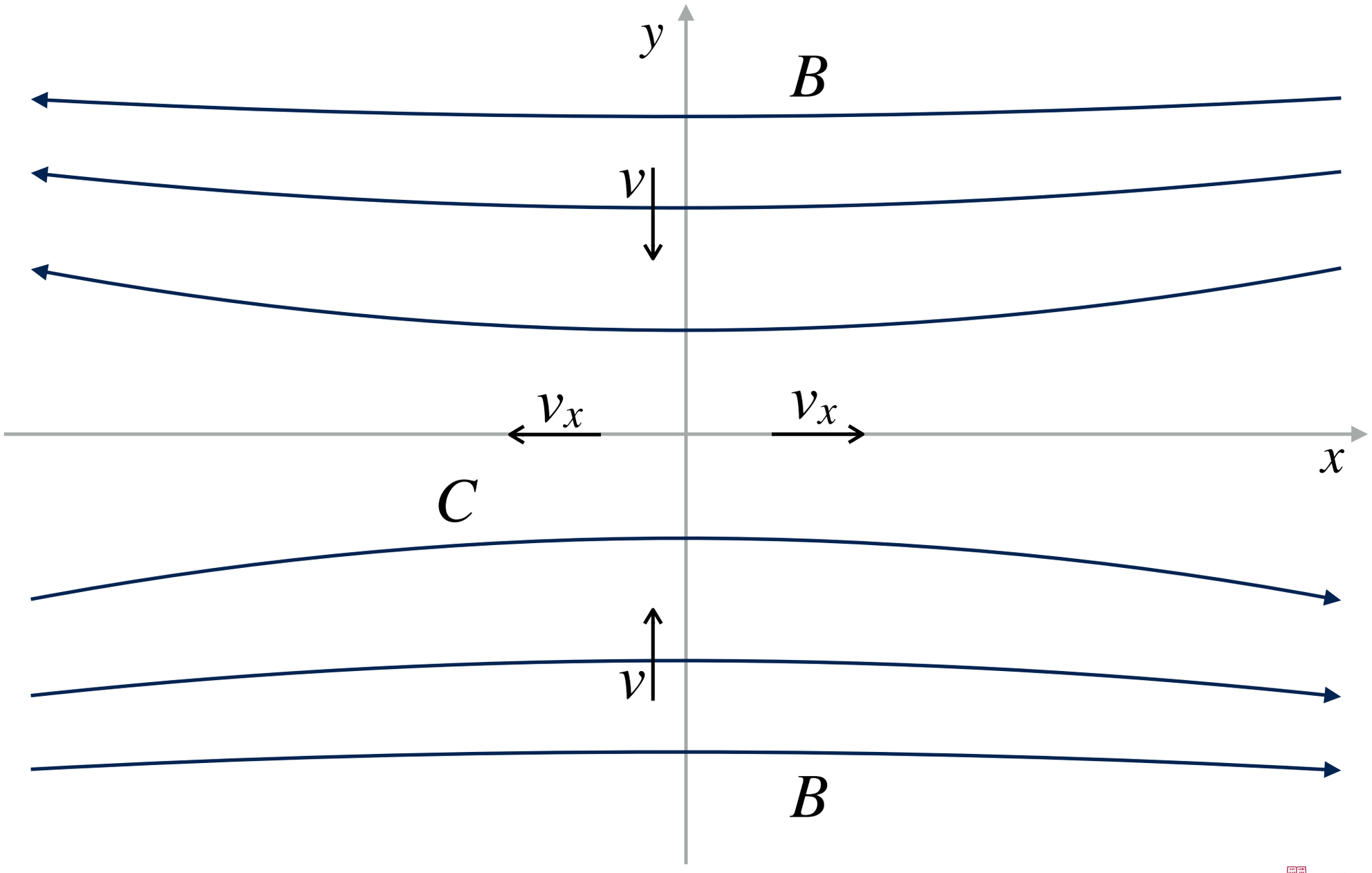
Magnetic Reconnection



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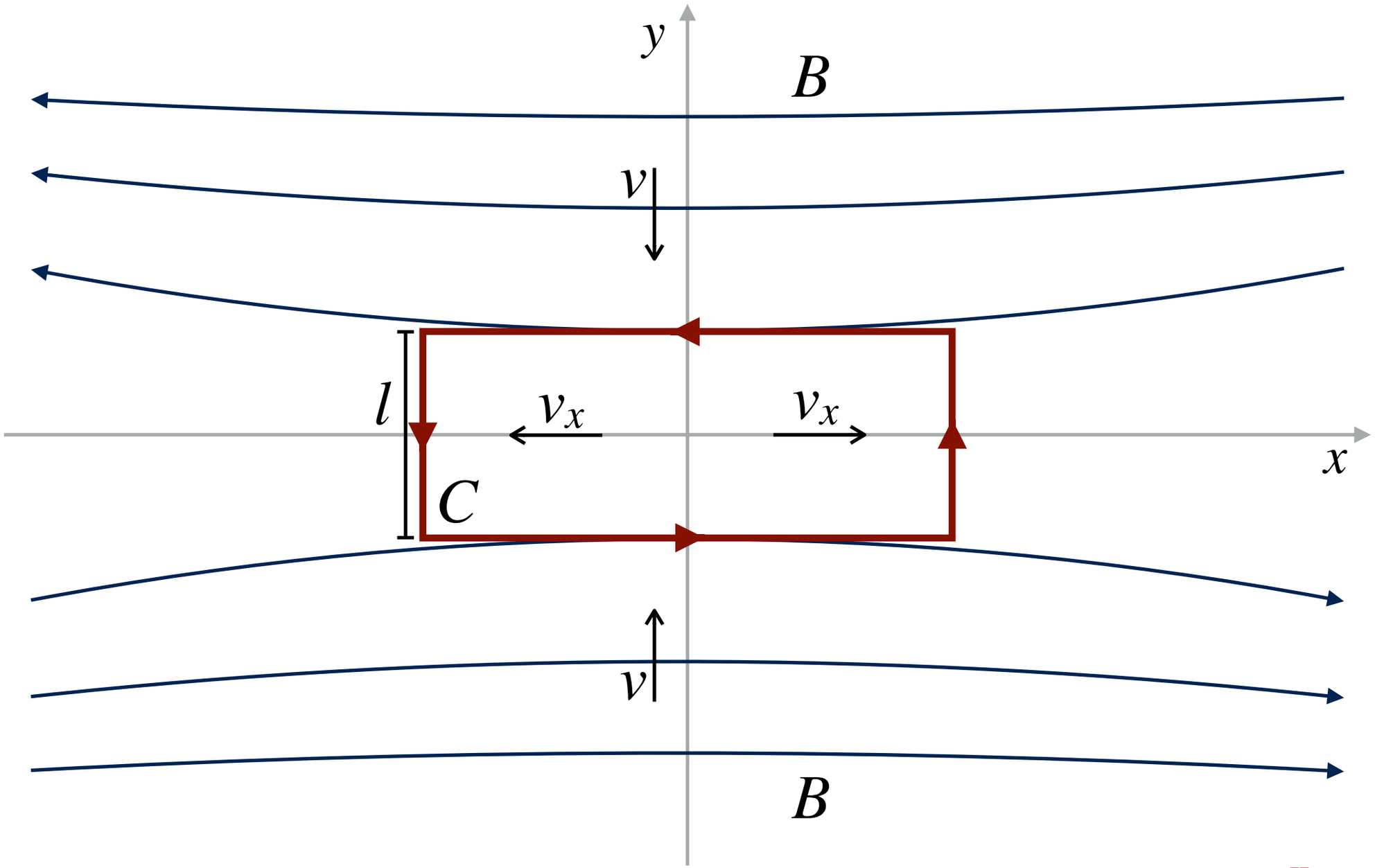
Magnetic Reconnection



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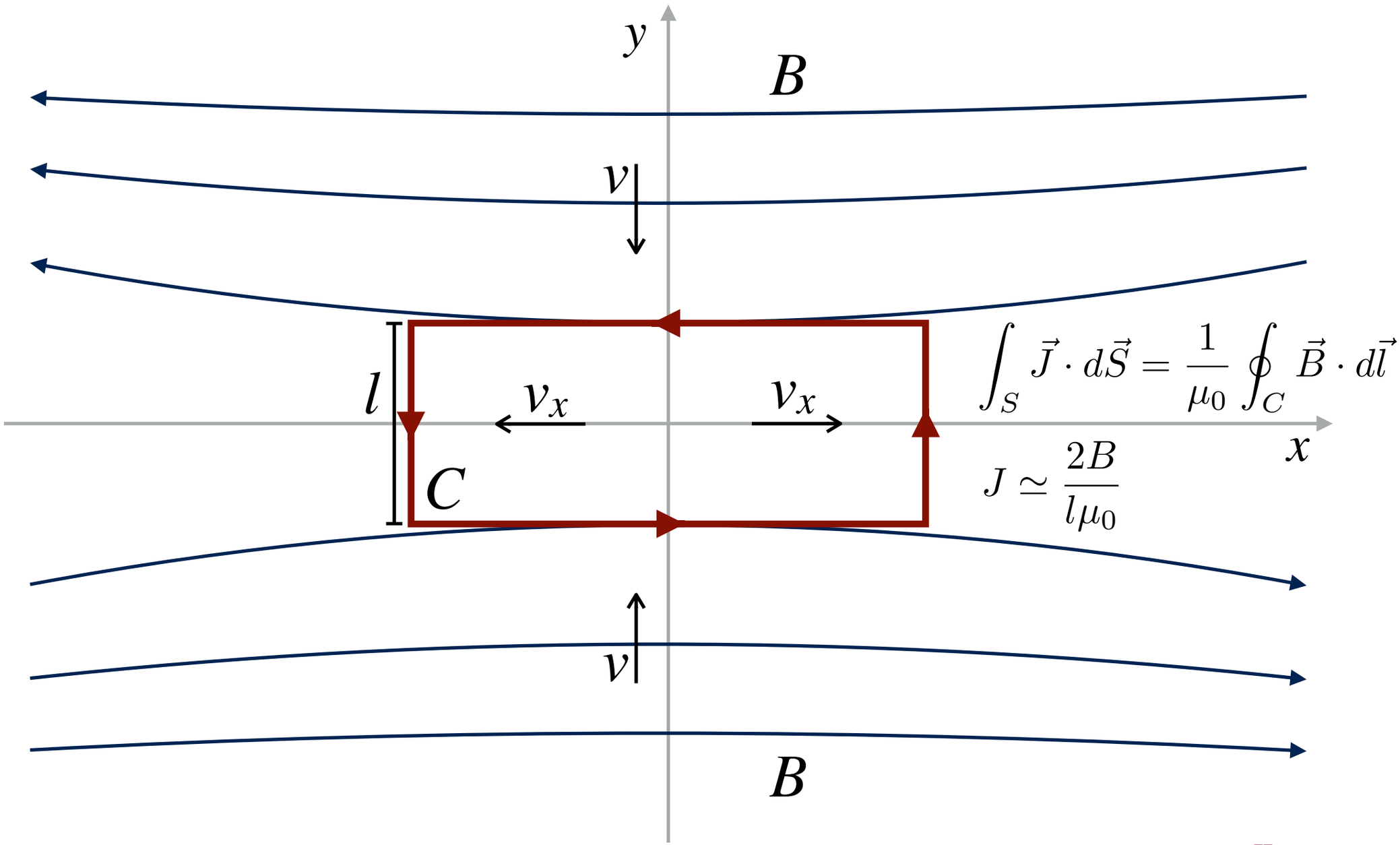
Magnetic Reconnection



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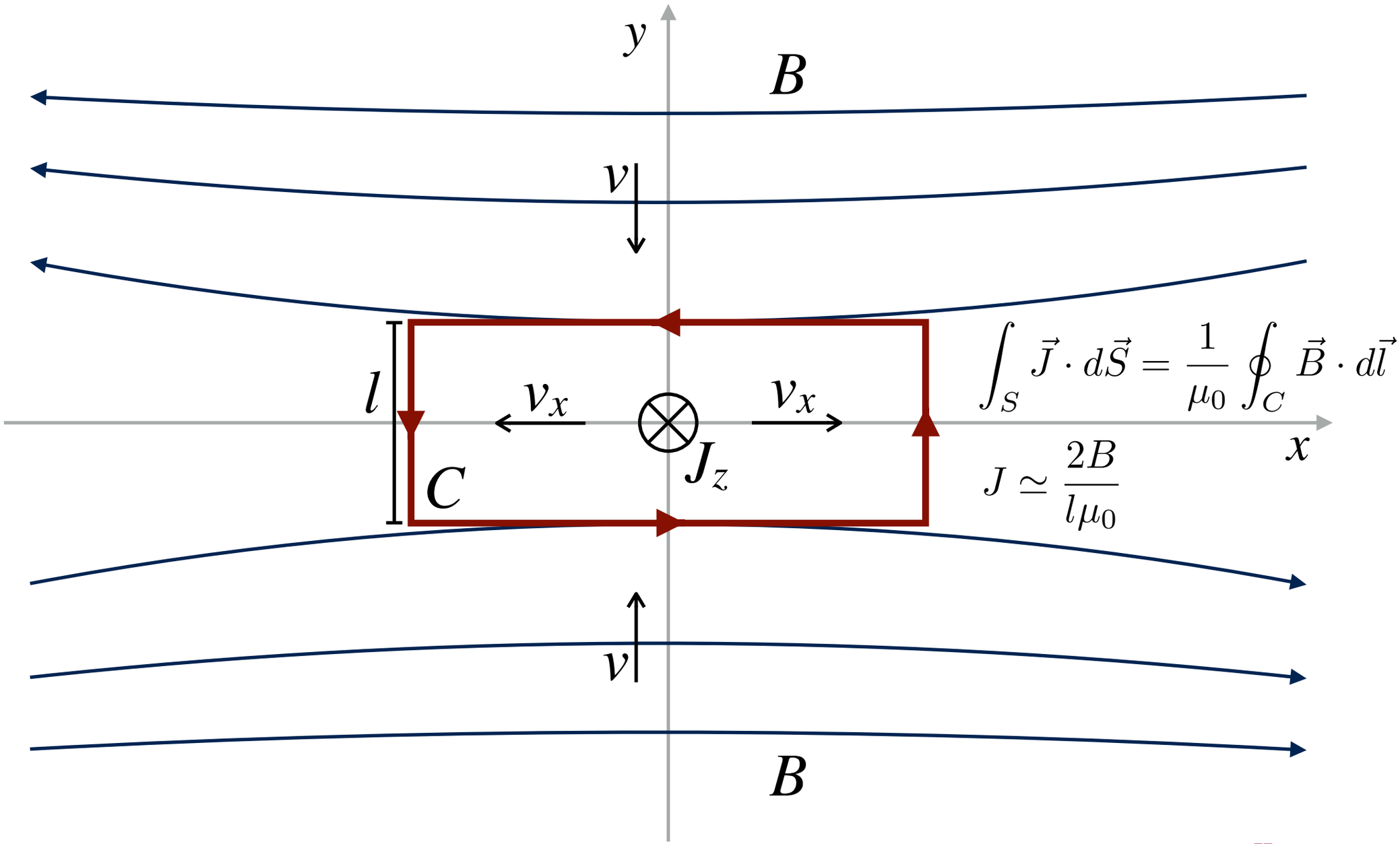
Magnetic Reconnection



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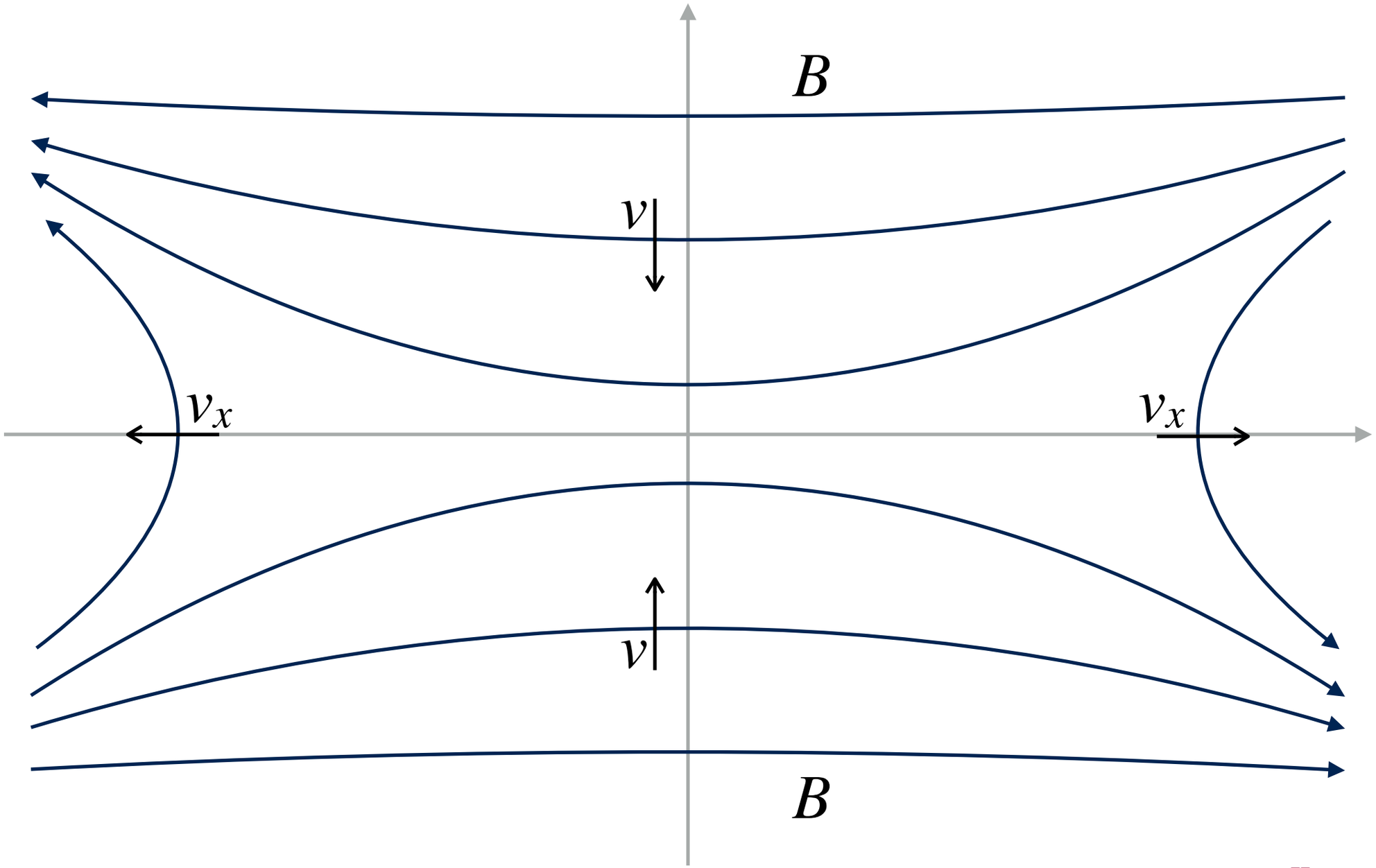
Magnetic Reconnection



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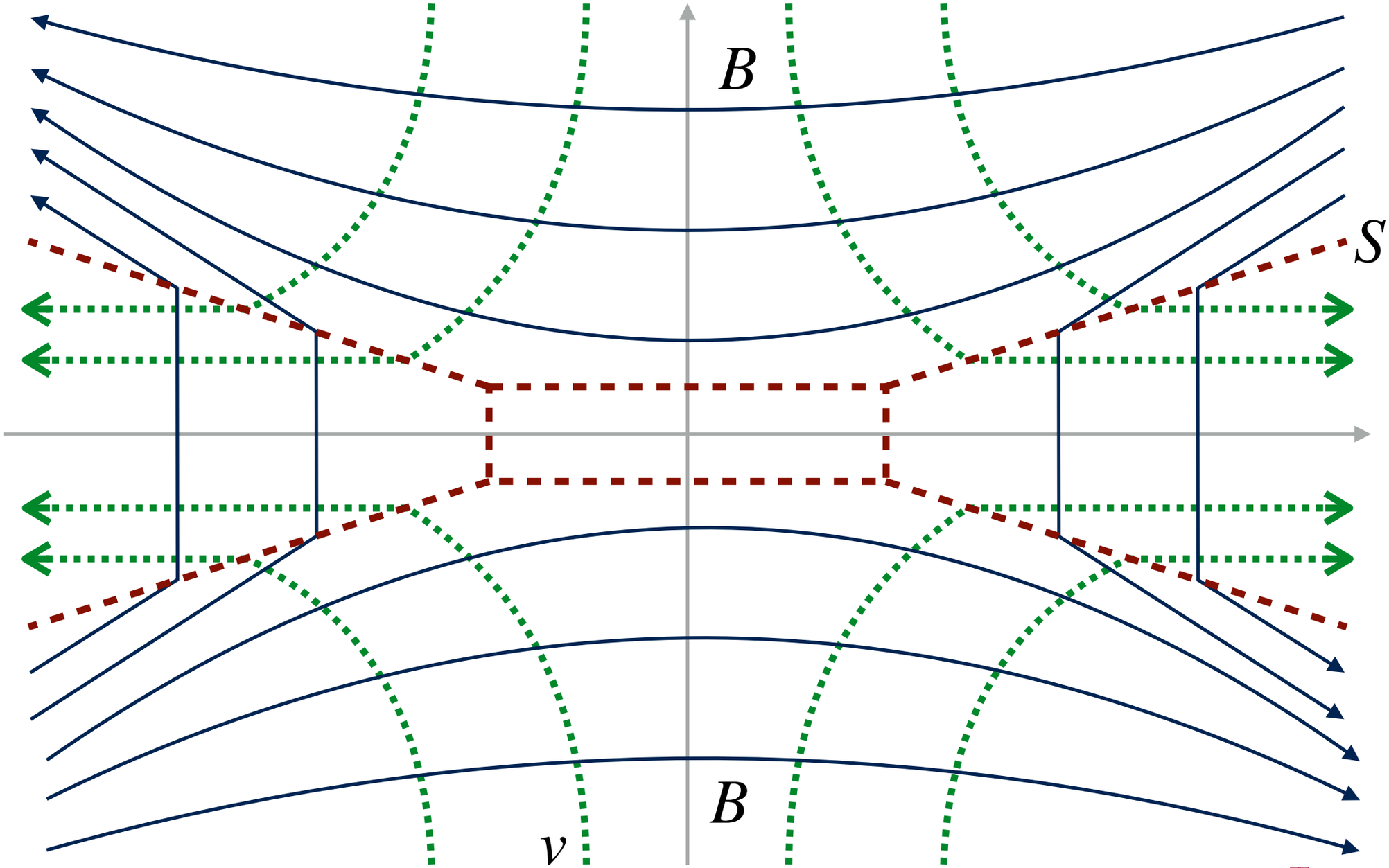
Magnetic Reconnection



Magnetic Field Fluid Flow Shock Front

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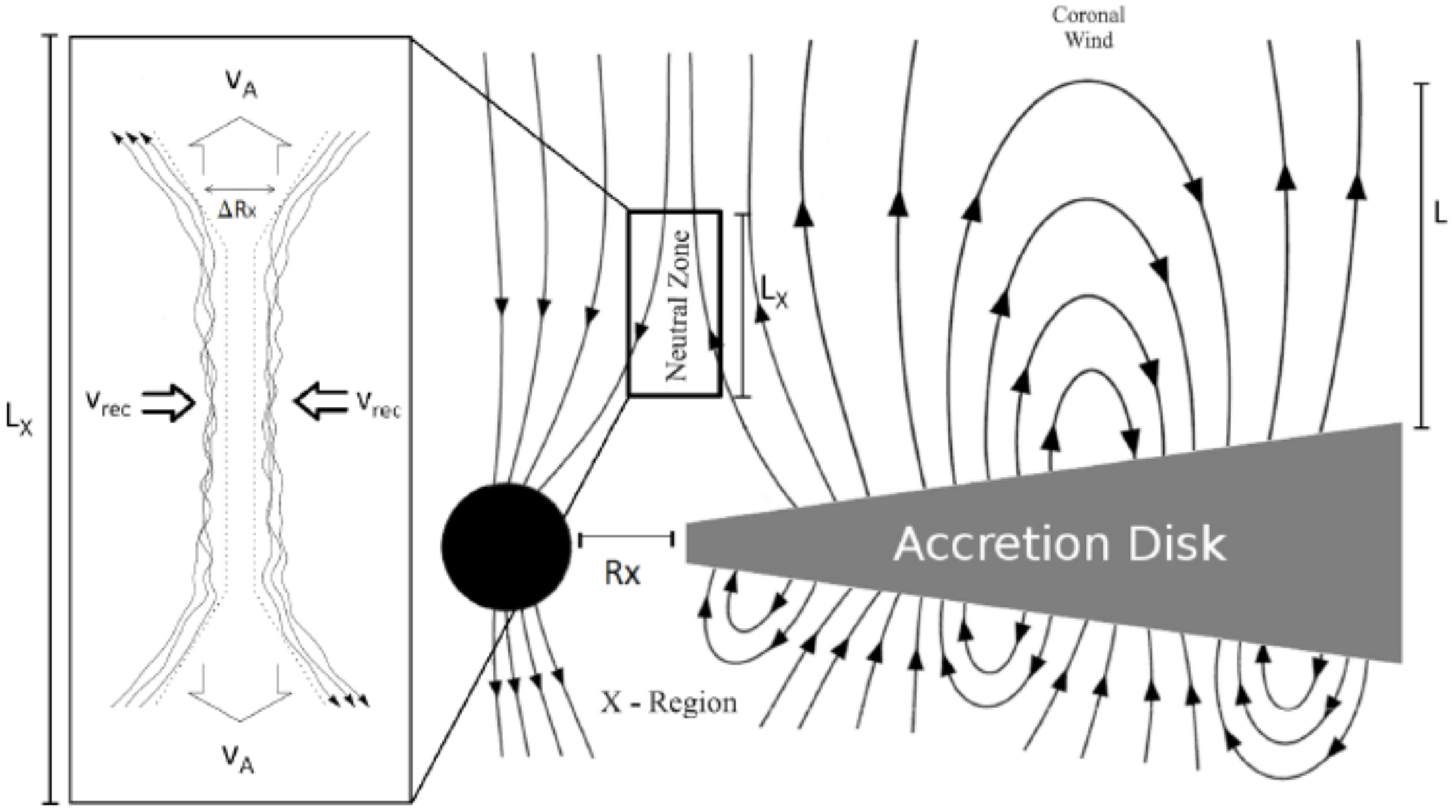

de Gouveia Dal Pino et al. 2016

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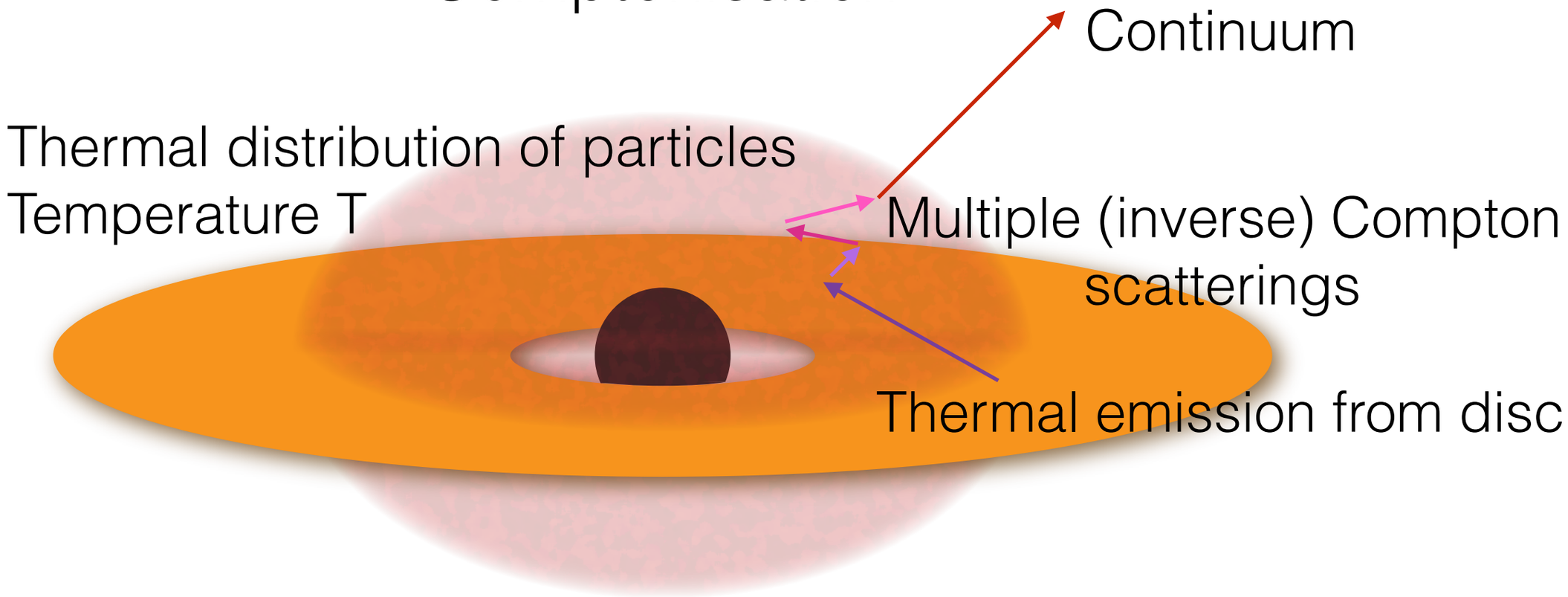
Producing an X-ray Continuum

Comptonisation



Average energy change of photon per Compton scattering

$$\left\langle \frac{\Delta\epsilon}{\epsilon} \right\rangle = \frac{4kT_e - \hbar\omega}{m_e c^2}$$

The Compton optical depth $y = \frac{kT_e}{m_e c^2} \max\left(\tau_e, \tau_e^2\right)$

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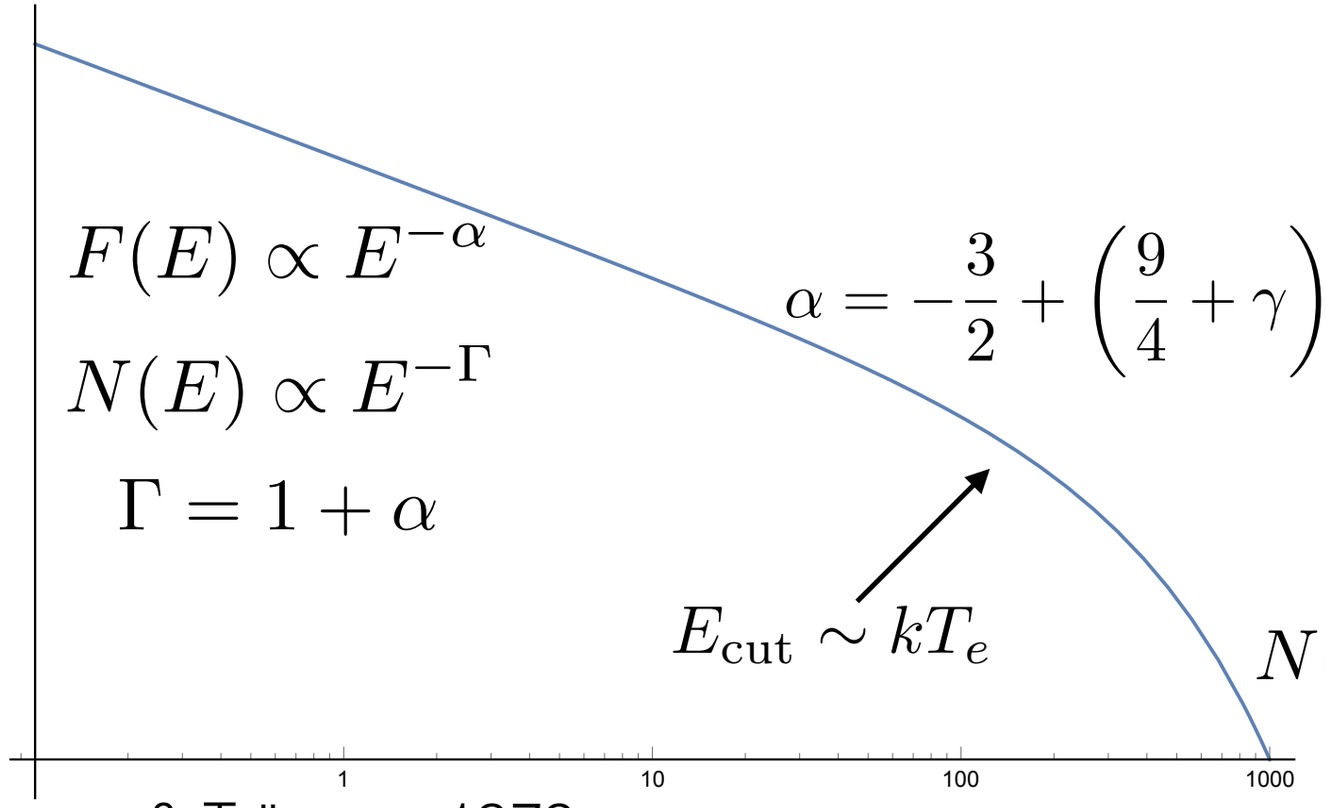
Producing an X-ray Continuum

Comptonisation

Resulting photon distribution is described by the Kompaneets equation (accounting for scattering as well as spontaneous and stimulated emission into each energy level)

$$\frac{\partial n}{\partial y} = \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(n + n^2 + \frac{\partial n}{\partial x}\right)\right] + Q(x) - \frac{n}{y_{\mathrm{escape}}} \qquad x = \frac{\hbar\omega}{kT_e}$$



$$F(E) \propto E^{-\alpha}$$

$$N(E) \propto E^{-\Gamma}$$

$$\Gamma = 1 + \alpha$$

$$\alpha = -\frac{3}{2} + \left(\frac{9}{4} + \gamma\right)^{\frac{1}{2}}$$

$$\gamma = -\frac{\pi^2}{3} + \frac{m_{\mathrm{e}}c^2}{kT_e\left(\tau + \frac{2}{3}\right)^2}$$

for spherical geometry

$$E_{\mathrm{cut}} \sim kT_e$$

$$N(E) \propto e^{-(E/E_{\mathrm{cut}})}$$

Sunyaev & Trümper 1979

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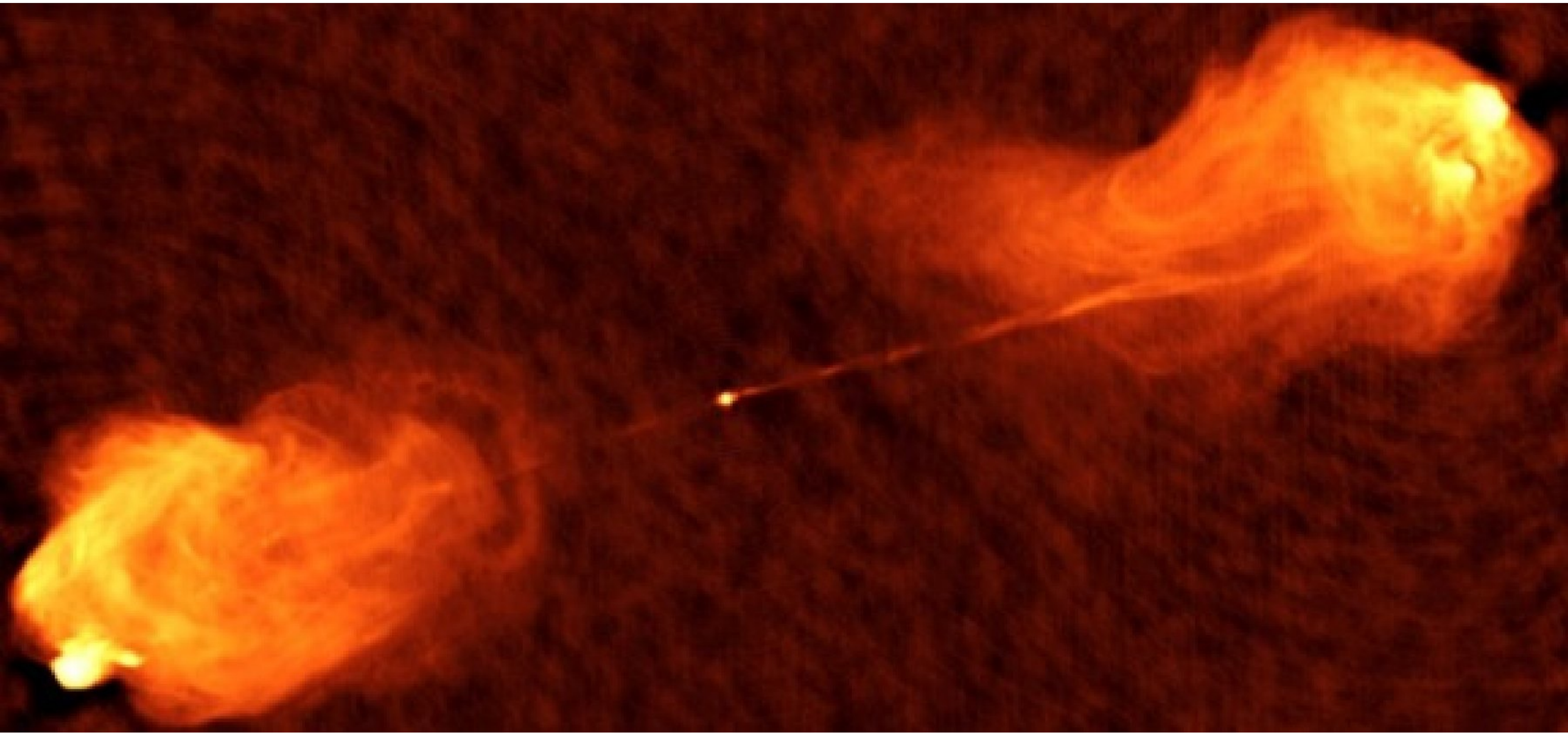

Jets

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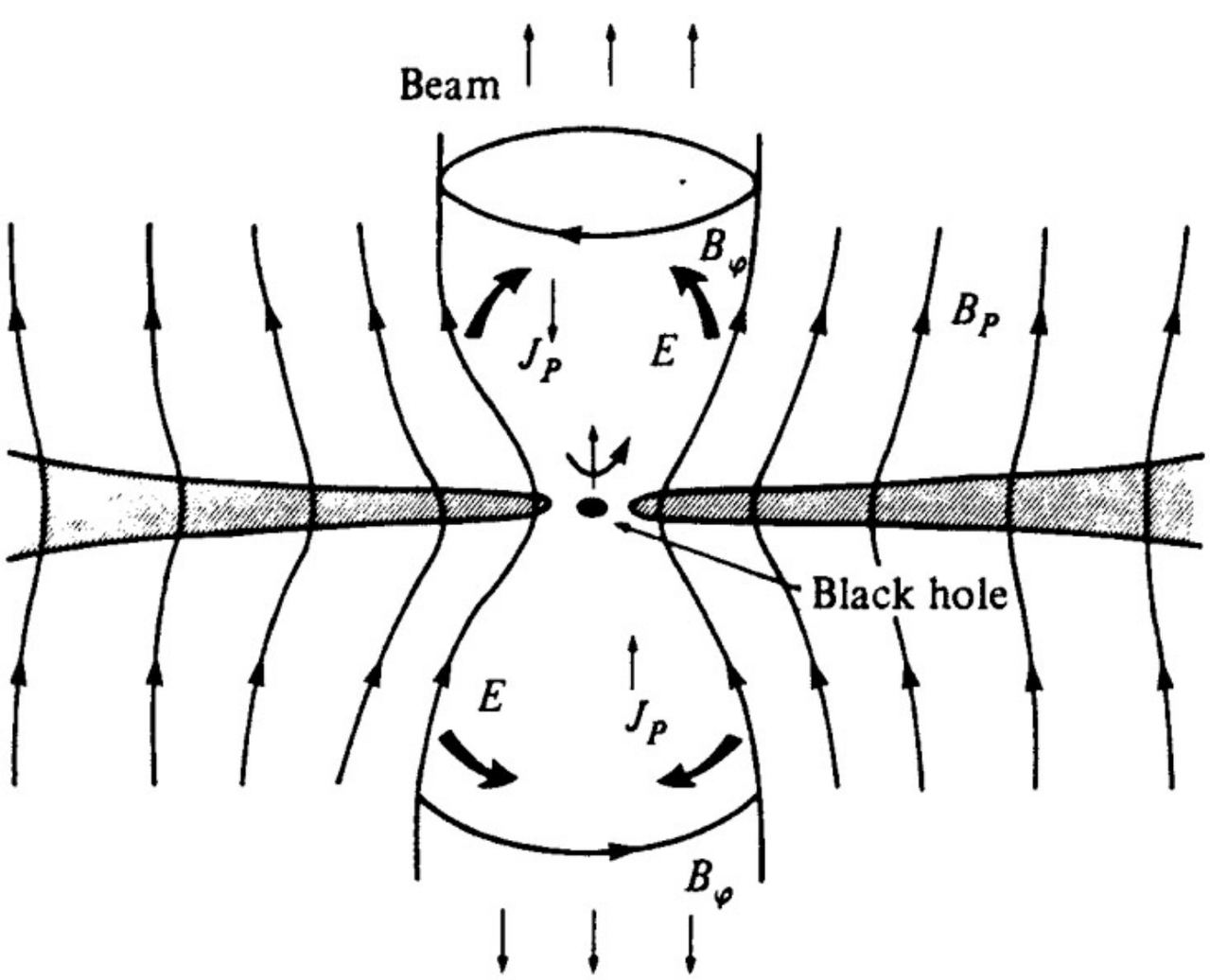
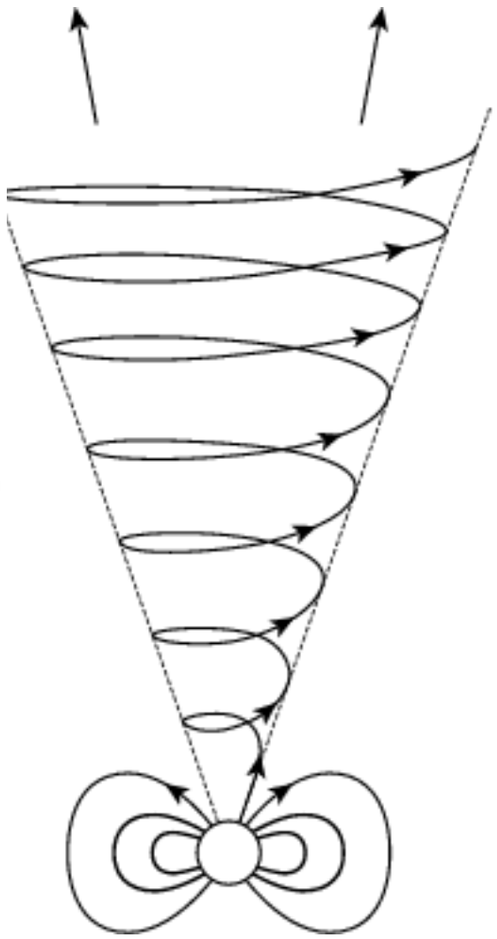
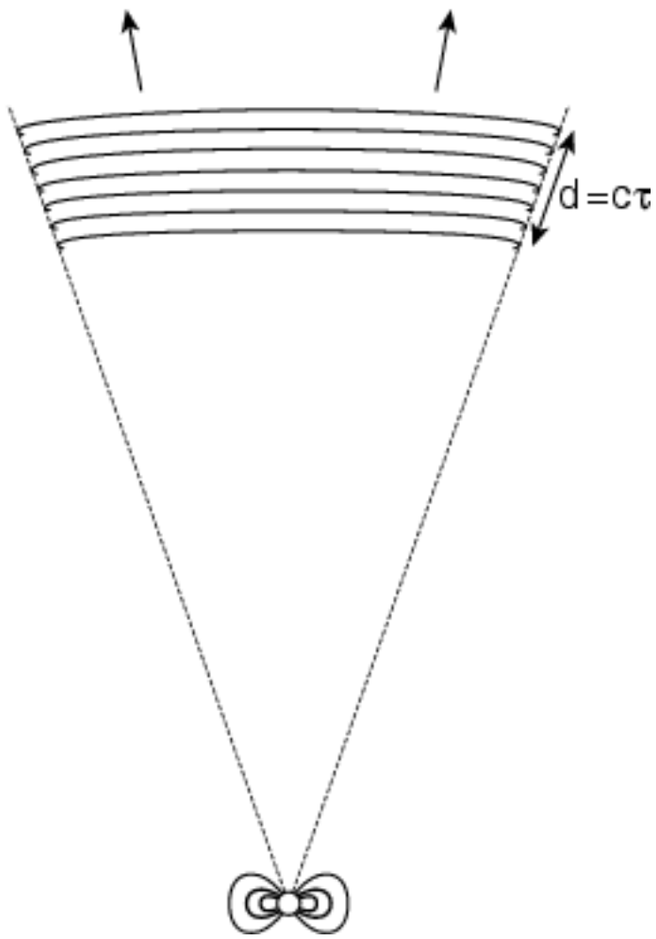
Magneto-centrifugal Jets







Blandford & Payne 1981

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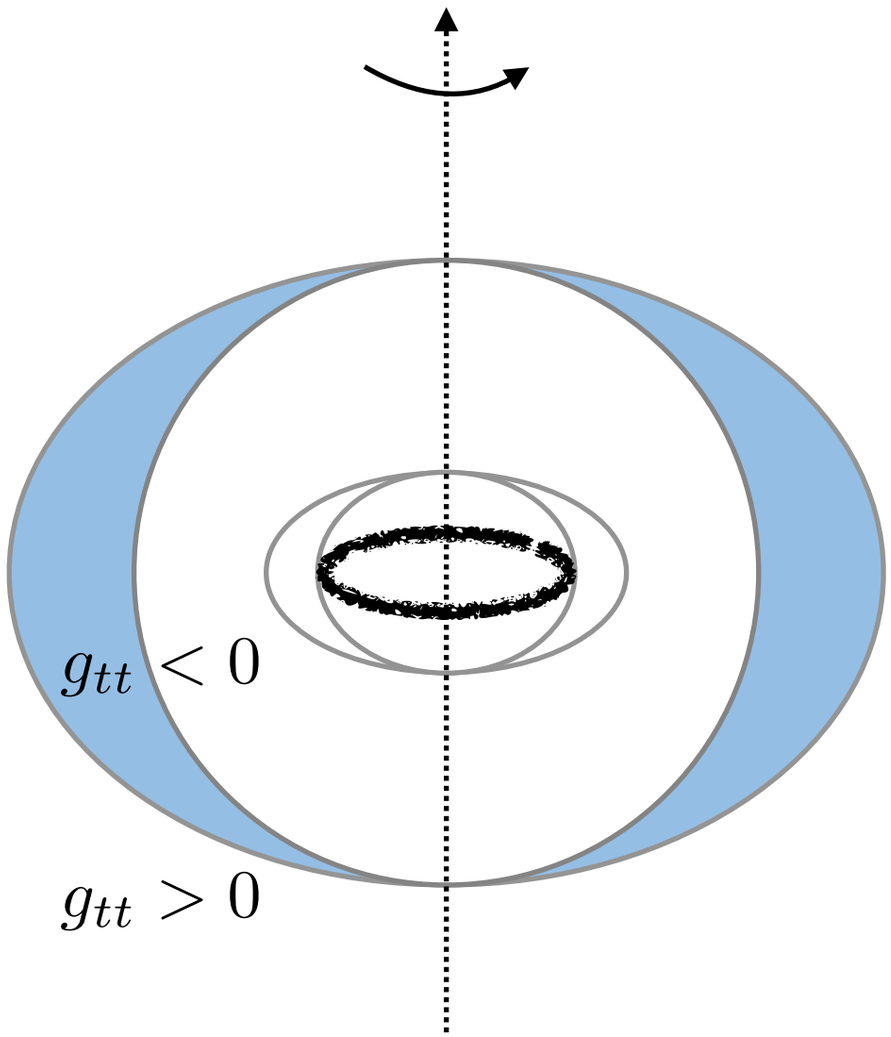
Extracting Energy from Black Holes

The Penrose Process



$$E^{(A)} = \mathbf{p}^{(A)}(\mathcal{E}) \cdot \mathbf{u} = p_t^{(A)}(\mathcal{E})$$

A

Penrose & Floyd 1971

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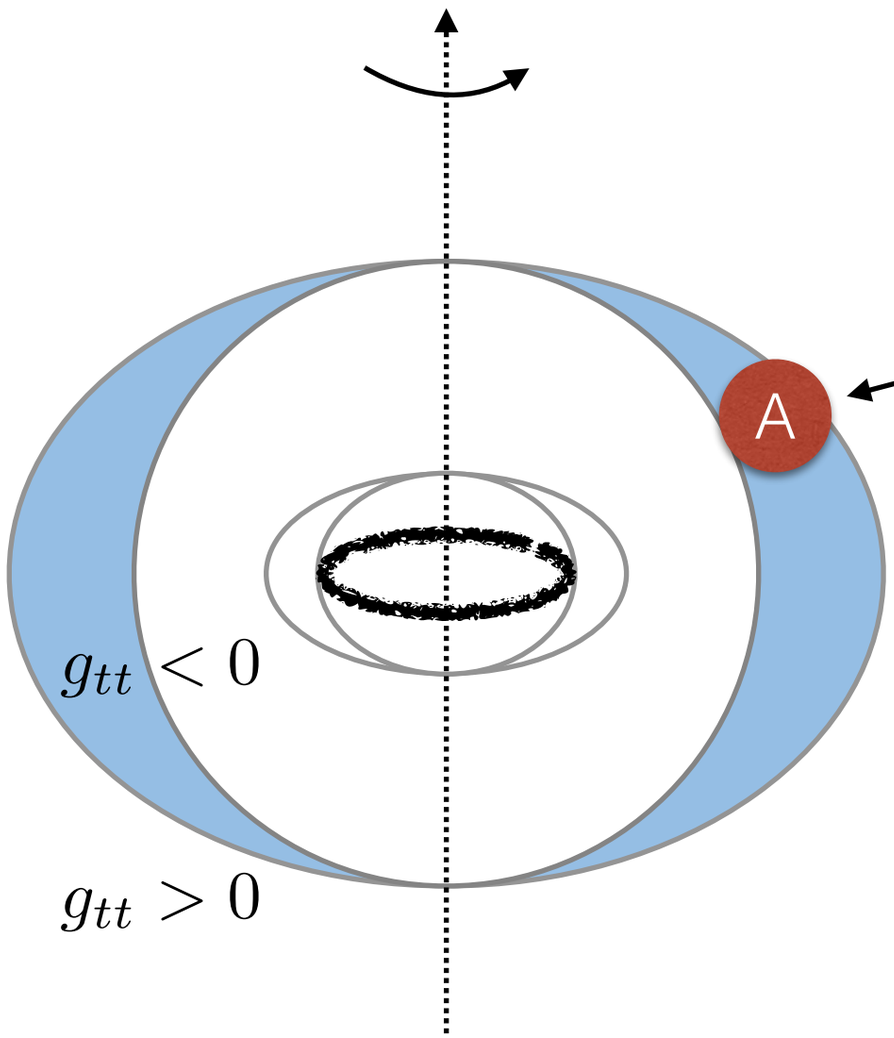

Penrose & Floyd 1971

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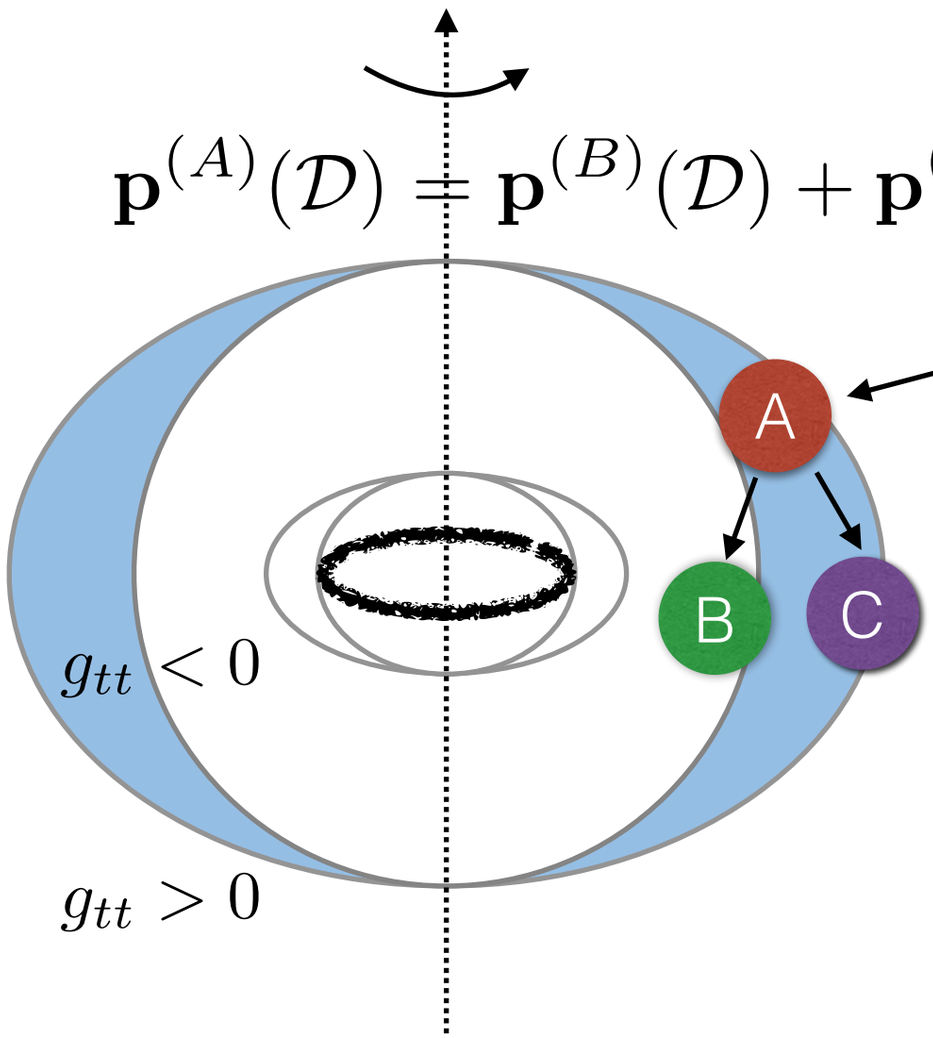
Extracting Energy from Black Holes

The Penrose Process

$$E^{(A)} = \mathbf{p}^{(A)}(\mathcal{E}) \cdot \mathbf{u} = p_t^{(A)}(\mathcal{E})$$

$$\mathbf{p}^{(A)}(\mathcal{D}) = \mathbf{p}^{(B)}(\mathcal{D}) + \mathbf{p}^{(C)}(\mathcal{D})$$



Penrose & Floyd 1971

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Extracting Energy from Black Holes

The Penrose Process

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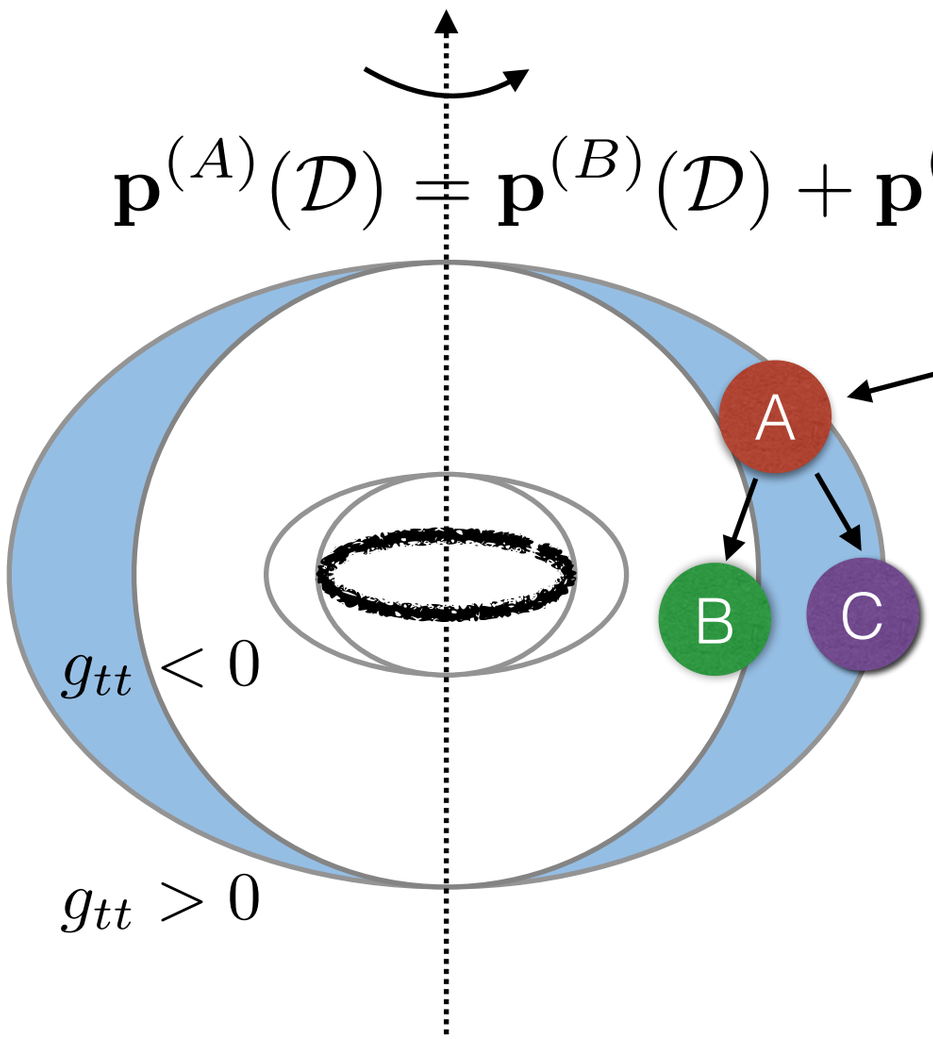

$$E^{(A)} = \mathbf{p}^{(A)}(\mathcal{E}) \cdot \mathbf{u} = p_t^{(A)}(\mathcal{E})$$

$$E^{(C)} = p_t^{(C)}(\mathcal{R}) = p_t^{(C)}(\mathcal{D})$$

Penrose & Floyd 1971

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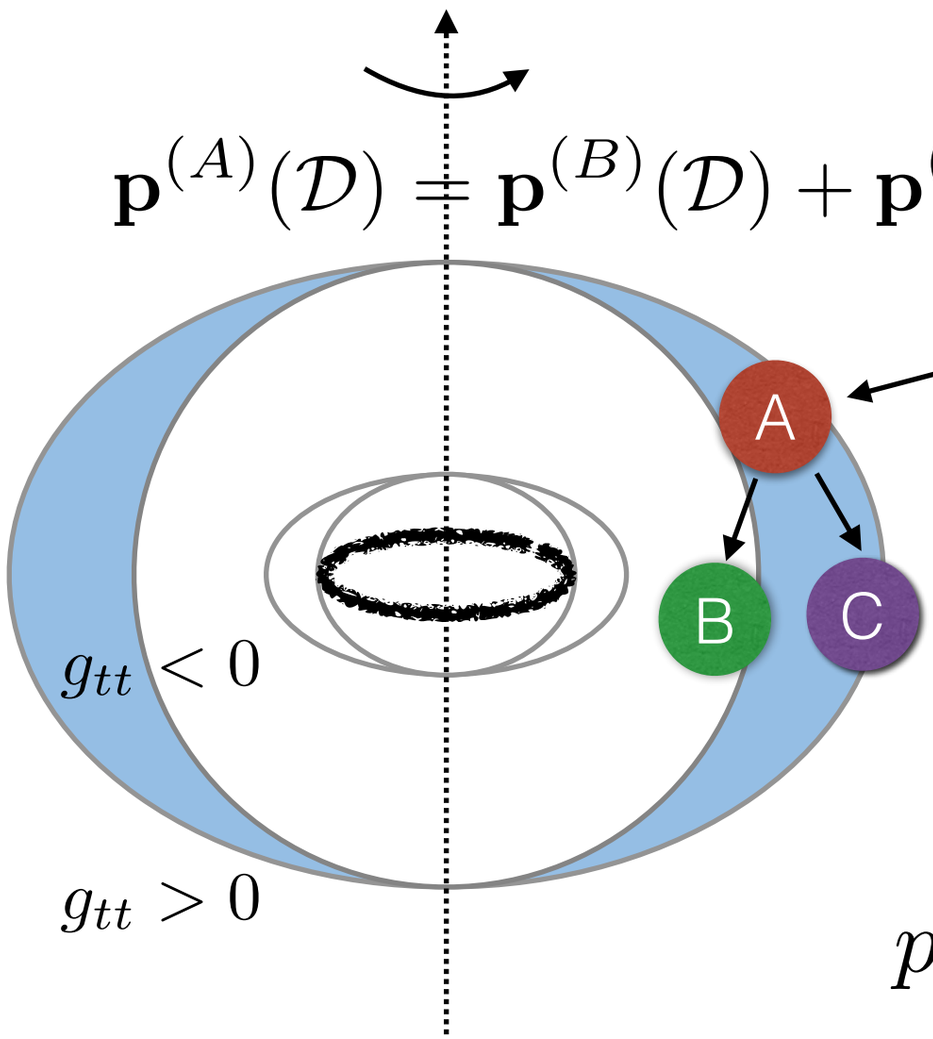
Extracting Energy from Black Holes

The Penrose Process



$$E^{(A)} = \mathbf{p}^{(A)}(\mathcal{E}) \cdot \mathbf{u} = p_t^{(A)}(\mathcal{E})$$

$$E^{(C)} = p_t^{(C)}(\mathcal{R}) = p_t^{(C)}(\mathcal{D})$$

$$E^{(C)} = E^{(A)} - p_t^{(B)}(\mathcal{D})$$

$$p_t^{(B)} = \mathbf{e}_t \cdot \mathbf{p}^{(B)} \text{ and } \mathbf{e}_t \cdot \mathbf{e}_t = g_{tt}$$

Penrose & Floyd 1971

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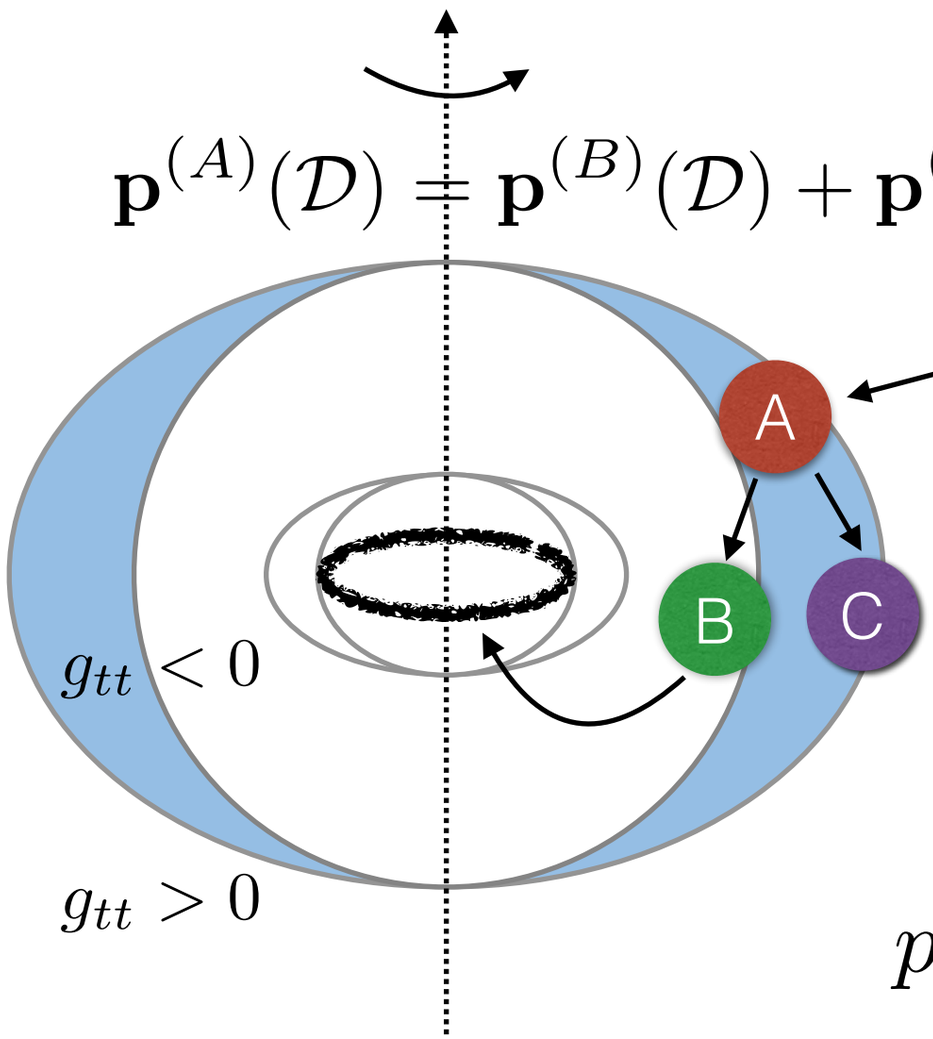
Extracting Energy from Black Holes

The Penrose Process

$$\mathbf{p}^{(A)}(\mathcal{D}) = \mathbf{p}^{(B)}(\mathcal{D}) + \mathbf{p}^{(C)}(\mathcal{D})$$



$$E^{(A)} = \mathbf{p}^{(A)}(\mathcal{E}) \cdot \mathbf{u} = p_t^{(A)}(\mathcal{E})$$

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$$E^{(C)} = E^{(A)} - p_t^{(B)}(\mathcal{D})$$

$$p_t^{(B)} = \mathbf{e}_t \cdot \mathbf{p}^{(B)} \text{ and } \mathbf{e}_t \cdot \mathbf{e}_t = g_{tt}$$

If $g_{tt} < 0$ (in ergoregion), possible that $p_t^{(B)} < 0$ (spacelike)

$$E^{(C)} > E^{(A)}$$

Penrose & Floyd 1971

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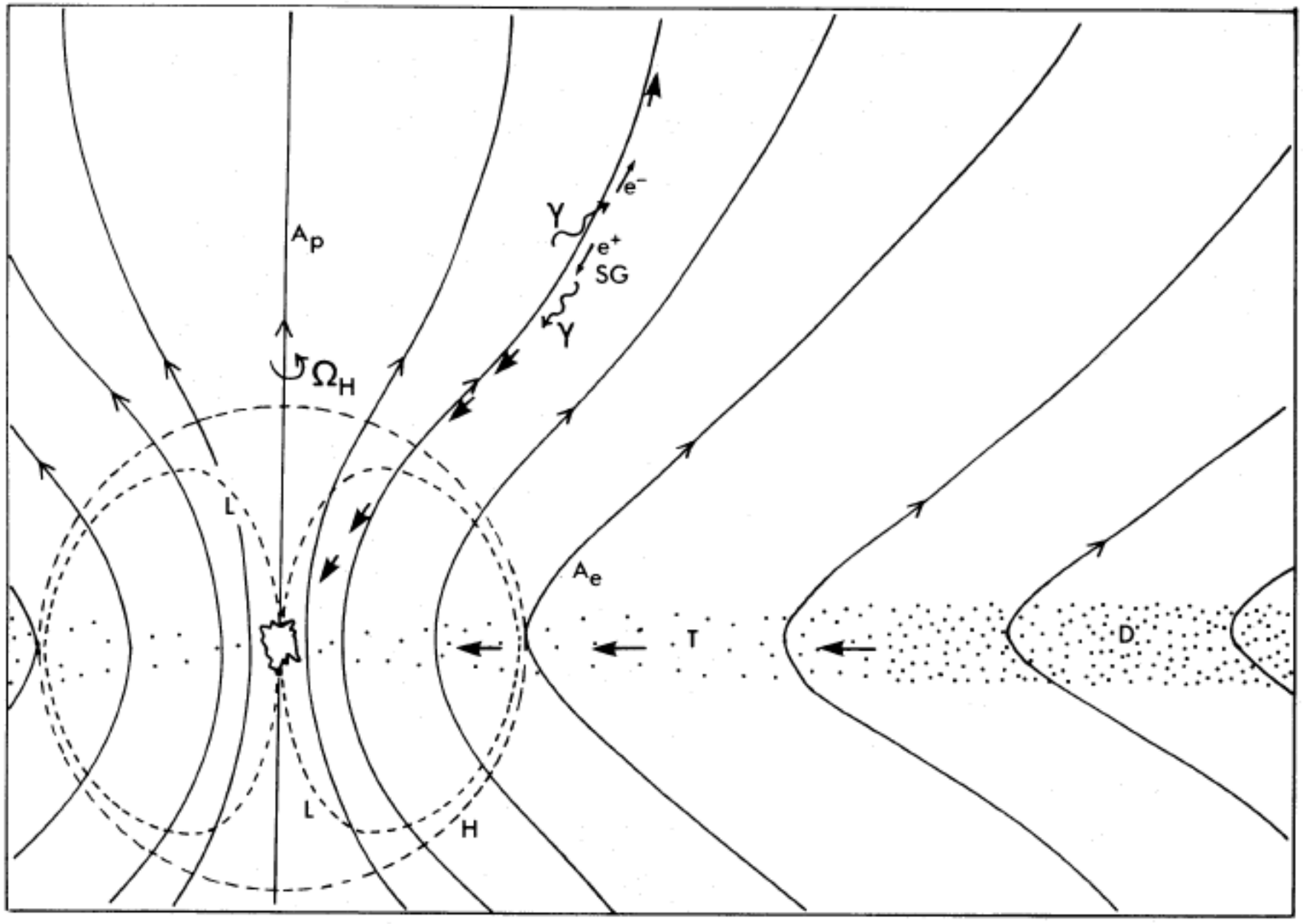
The Blandford-Znajek Process



Blandford & Znajek 1976

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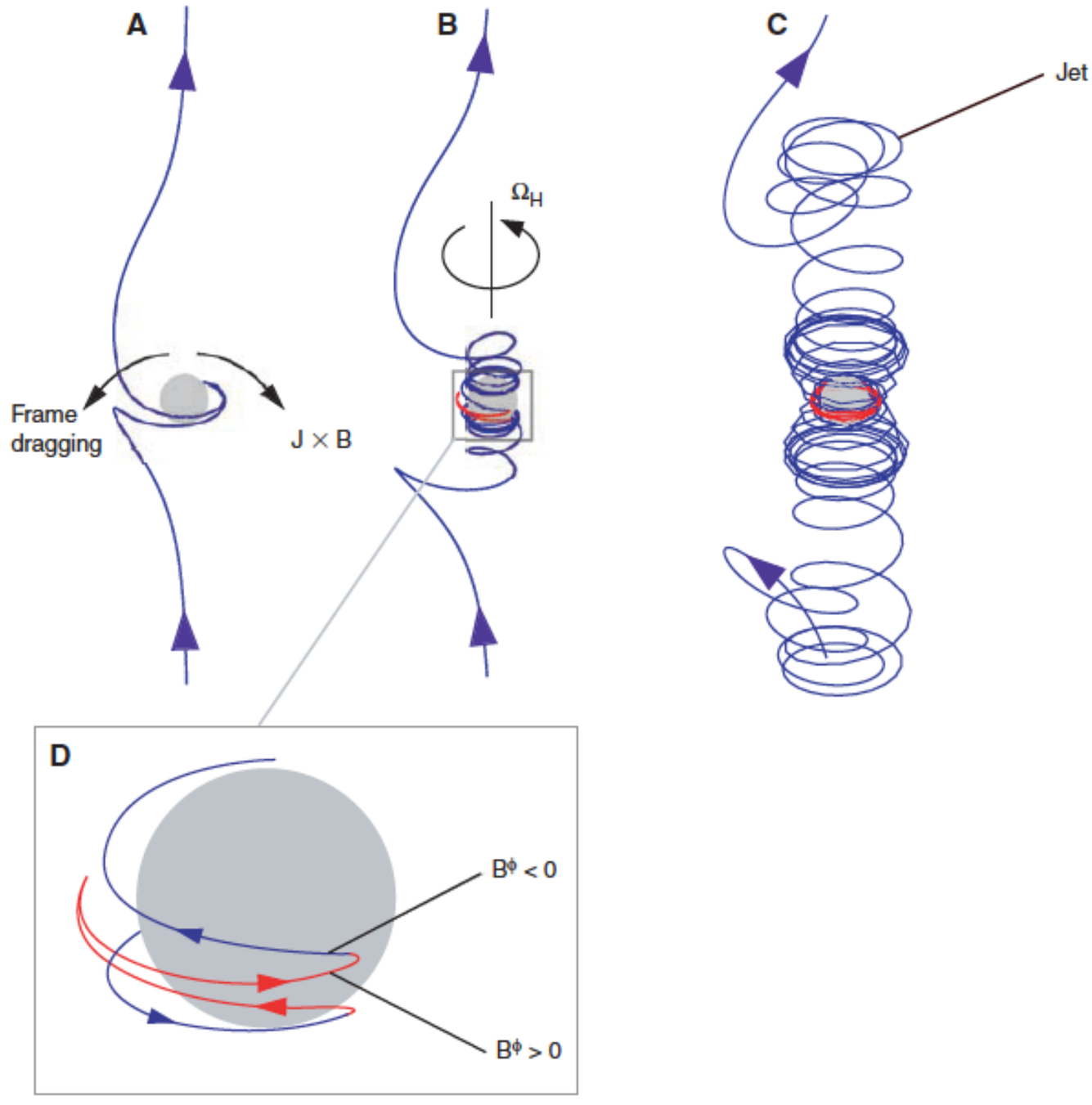
The Blandford-Znajek Process



Narayan & Quataert 2005

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State of the Art: GRMHD

- Need a covariant description of the electromagnetic field
- Lorentz force, $\vec{f} = q(\vec{E} + \vec{u} \wedge \vec{B})$ can be written

$$\mathbf{f} = q\mathbf{F} \cdot \mathbf{u} \qquad f_\mu = qF_{\mu\nu}u^\nu$$

- The EM field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(define A to reproduce E- and B-fields in a Cartesian frame)

$$[F_{\mu\nu}] = \begin{bmatrix} 0 & E^x/c & E^y/c & E^z/c \\ -E^x/c & 0 & -B^z & B^y \\ -E^y/c & B^z & 0 & -B^x \\ -E^z/c & -B^y & B^x & 0 \end{bmatrix}$$

- Maxwell's equations recovered from

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \qquad \partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\nu} + \partial_\mu F_{\nu\sigma} = 0$$

e.g. Gammie, McKinney & Tóth 2003

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GRMHD

- In the presence of an EM field, toe total stress-energy tensor (including gas pressure and internal energy)

$$T_{\mu\nu} = T^{\rm Mat}_{\mu\nu} + T^{\rm EM}_{\mu\nu}$$

$$T^{\rm Mat}_{\mu\nu} = (\rho_0 + u_g + p_g)u_\mu u_\nu + p_g \delta_{\mu\nu}$$

$$T^{\rm EM}_{\mu\nu} = \frac{1}{\mu_0}\left(F_{\mu\sigma}F^{\sigma}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)$$

- The geodesic equations (and evolution of the field) come from

$$\nabla_\mu T^\mu_\nu = 0$$

e.g. Gammie, McKinney & Tóth 2003

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Simulation Diagnostics

- Mass flux (accretion rate)
- Energy/angular momentum flux (from $T_{\mu\nu}$)
- Magnetic flux (averaged vs. radius or angle)
- Disc thickness (based on optical depth or density)

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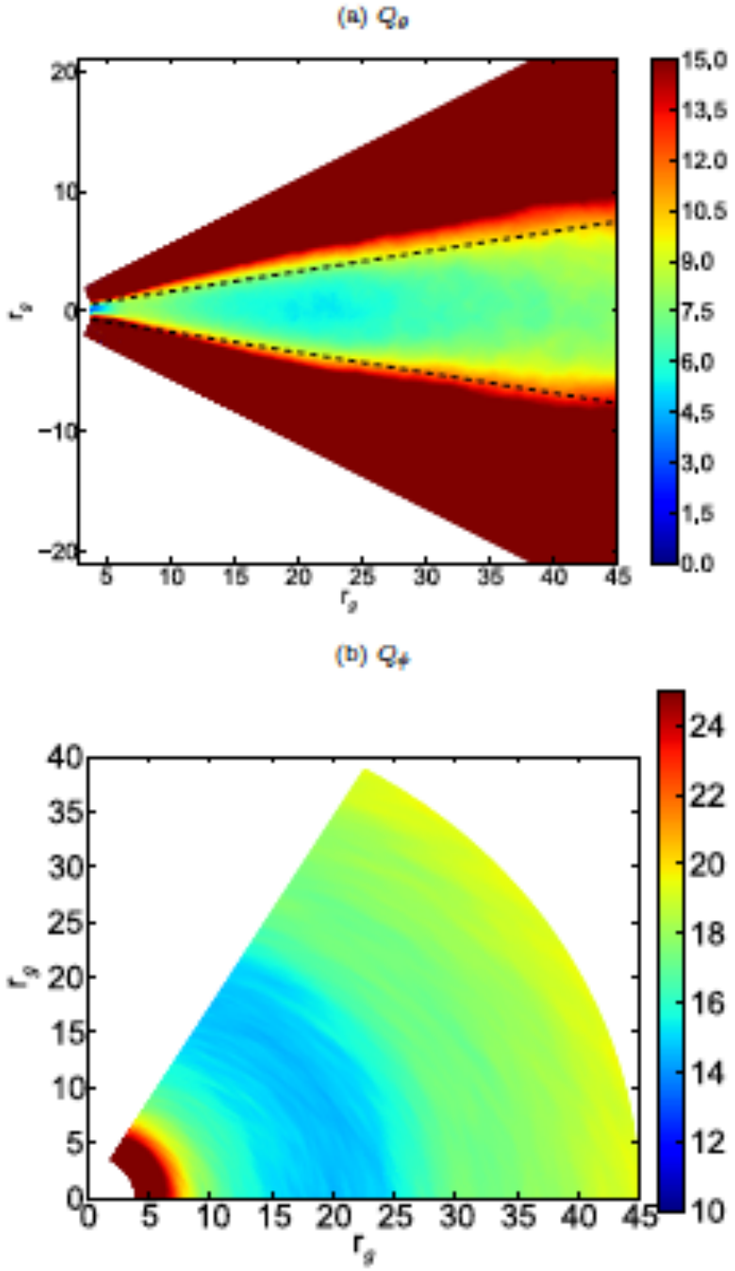

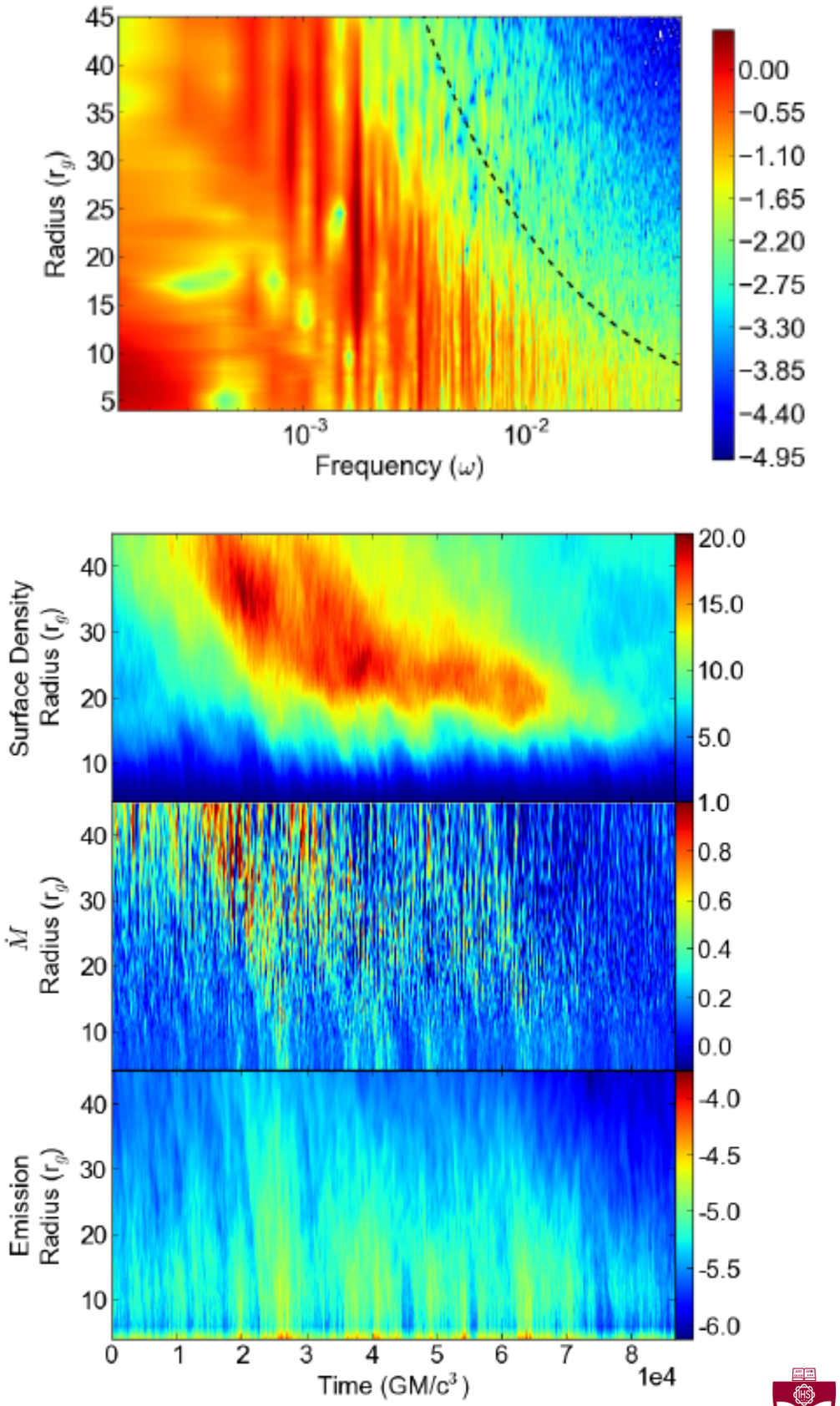

Hogg & Reynolds 2016

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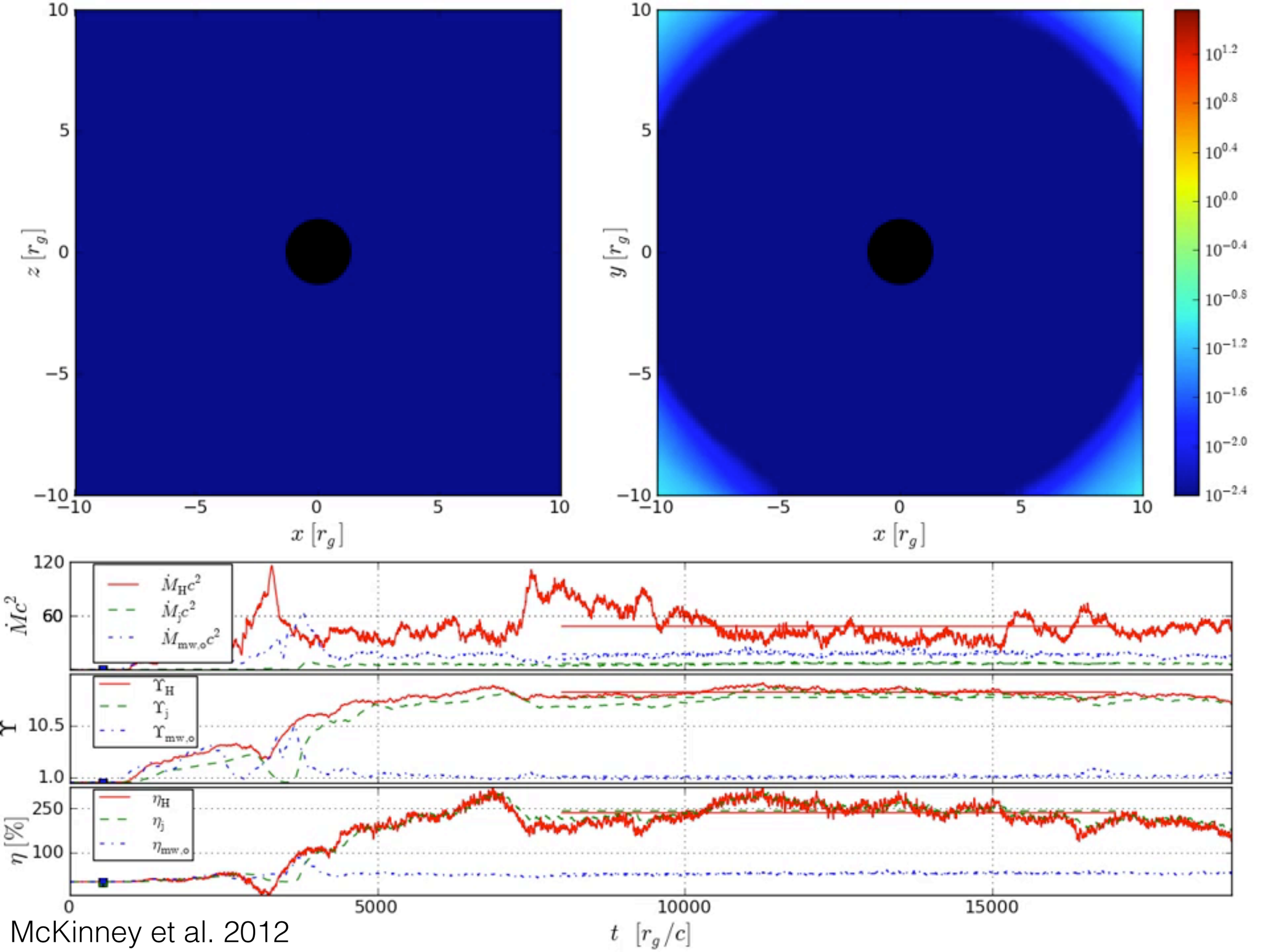
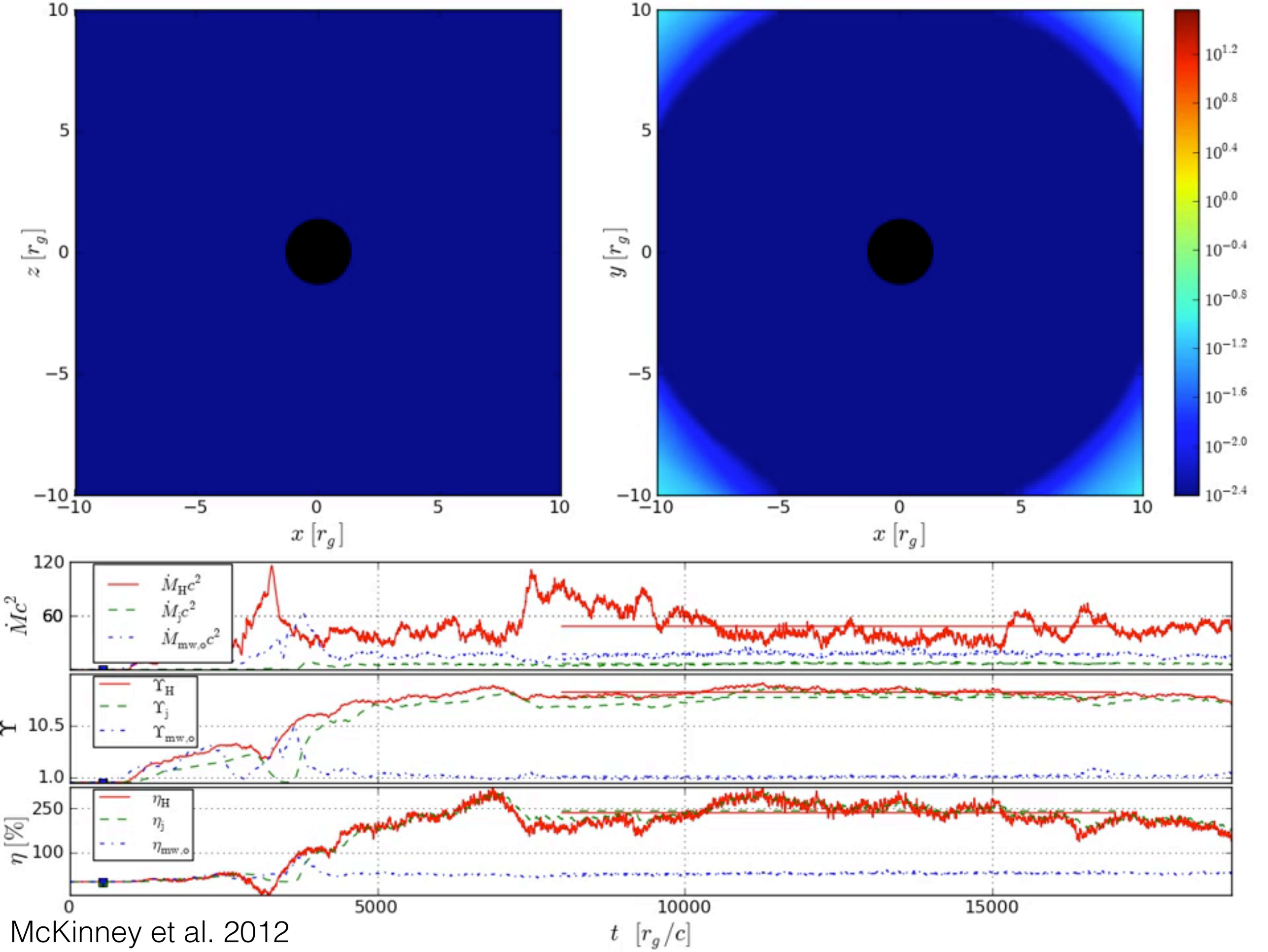
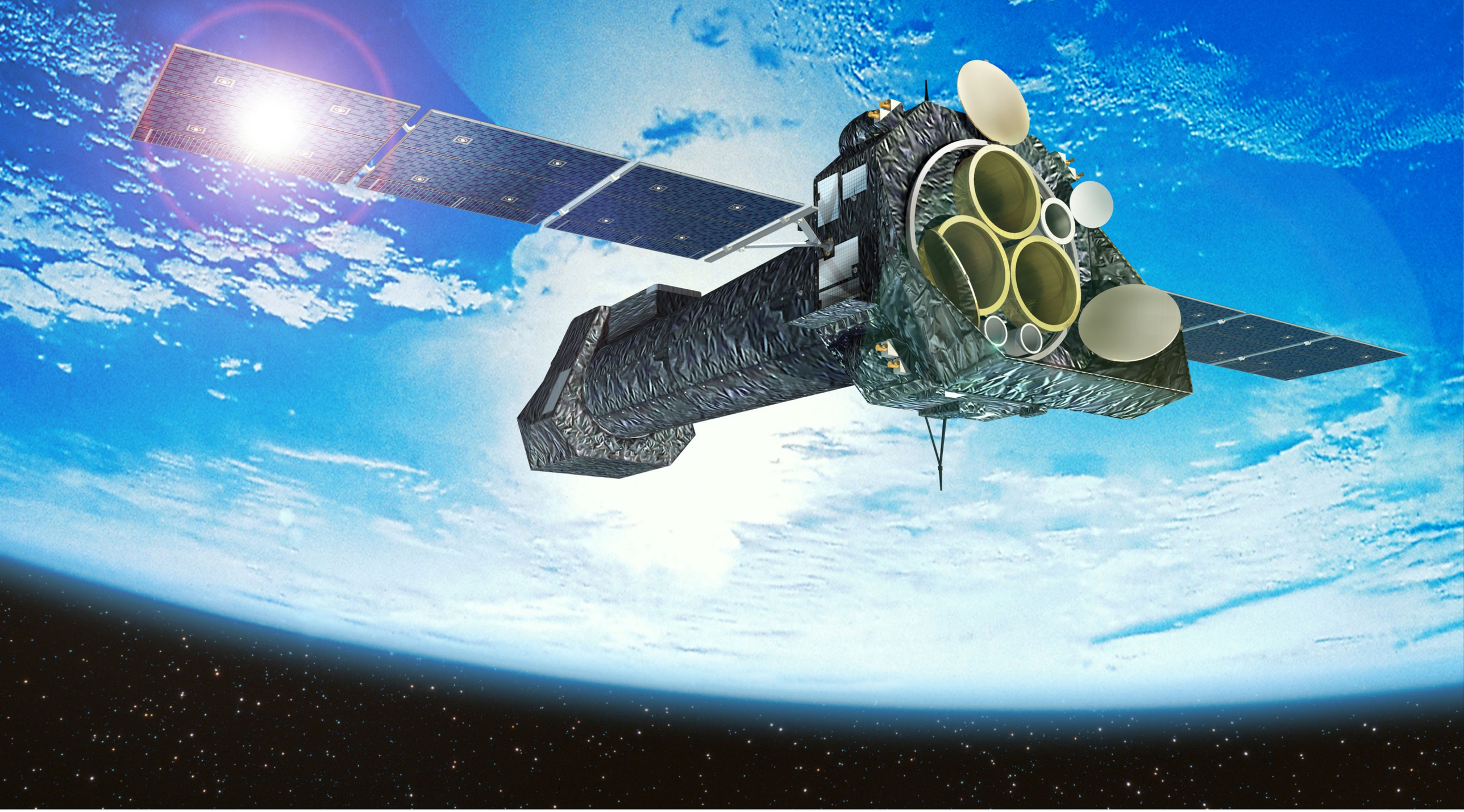


McKinney et al. 2012



McKinney et al. 2012



Using the theory to get close to black holes

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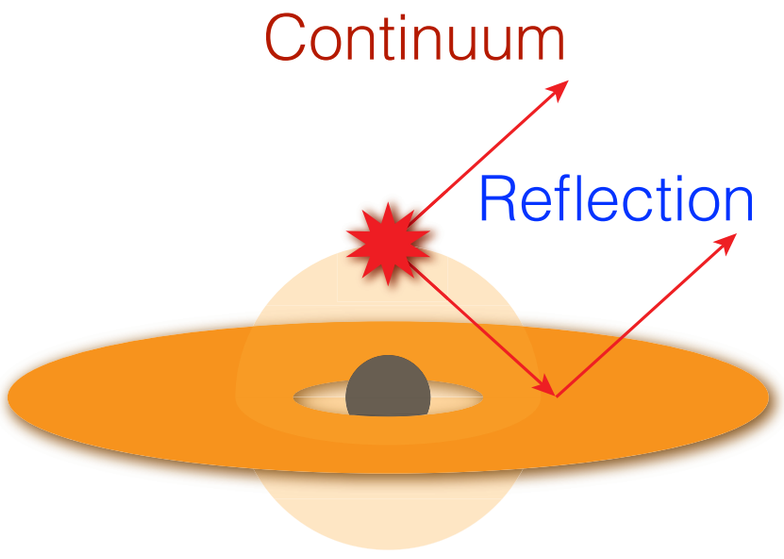
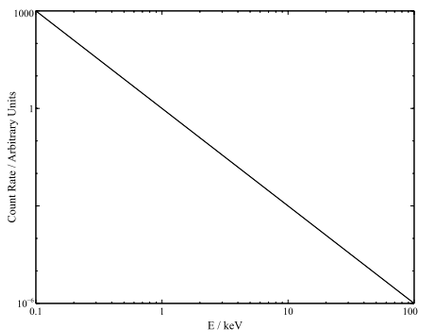
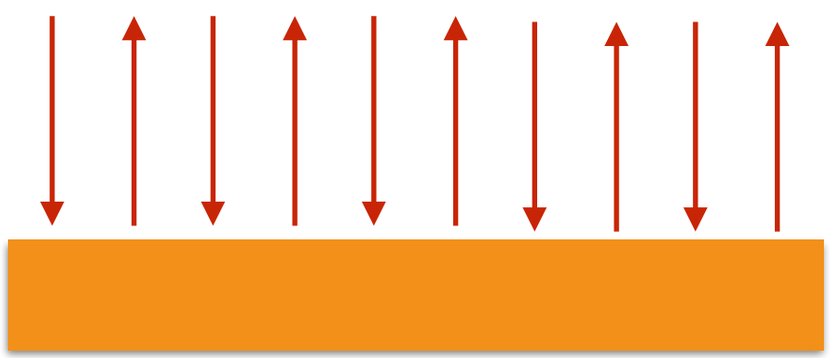
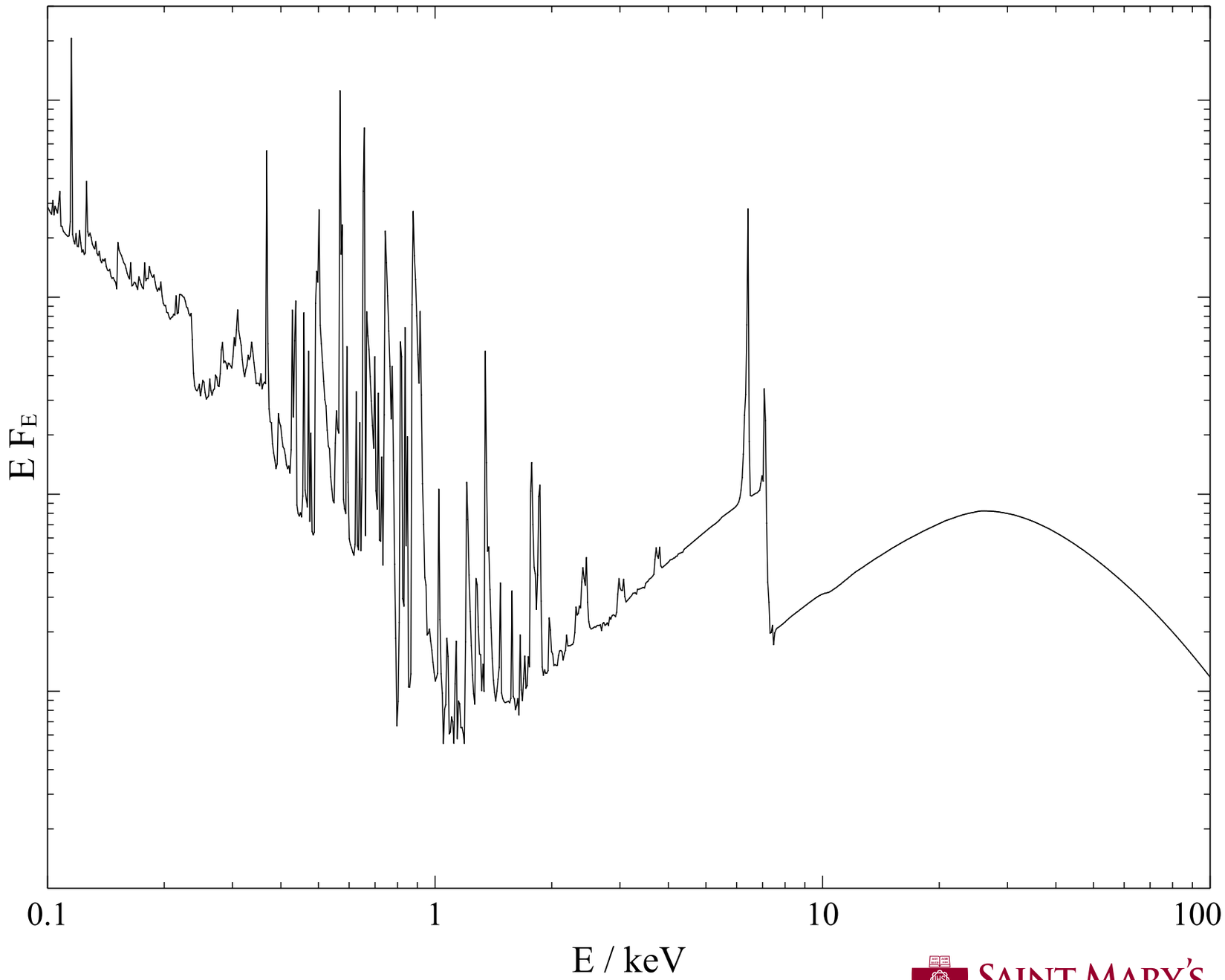
Reflection from Accretion Discs



Irradiate disc with power law







Ross & Fabian 2005, García et al. 2010

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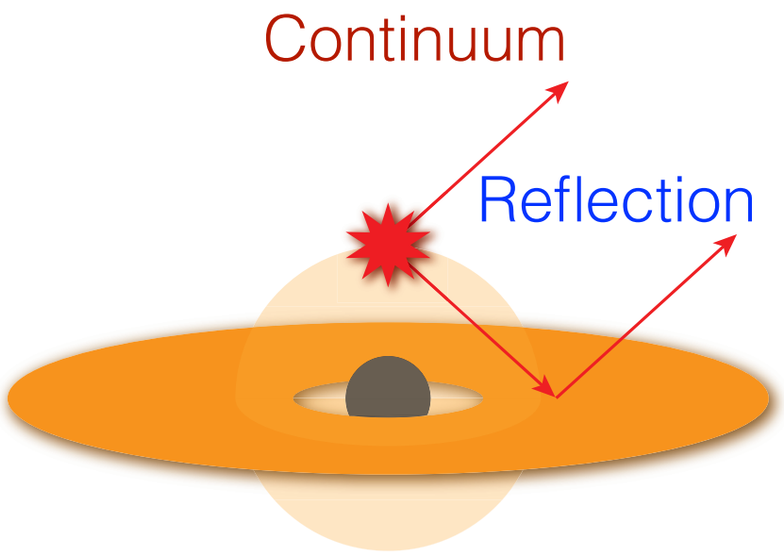
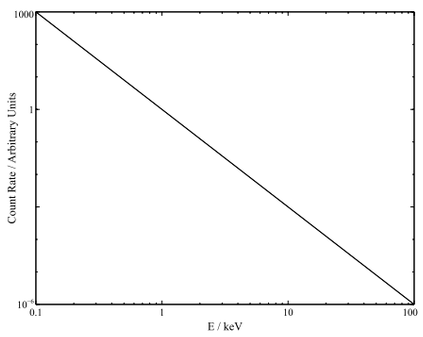
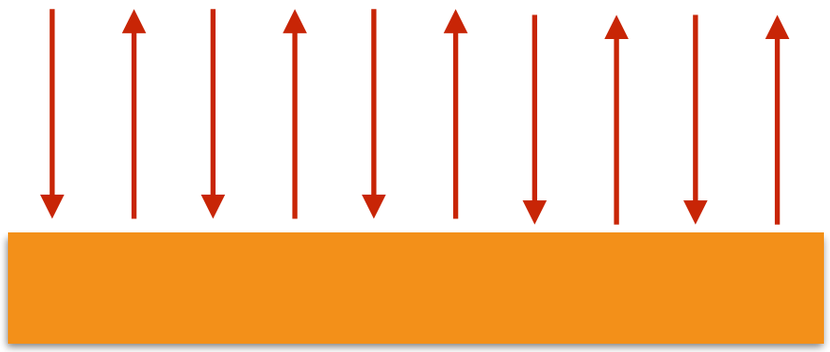
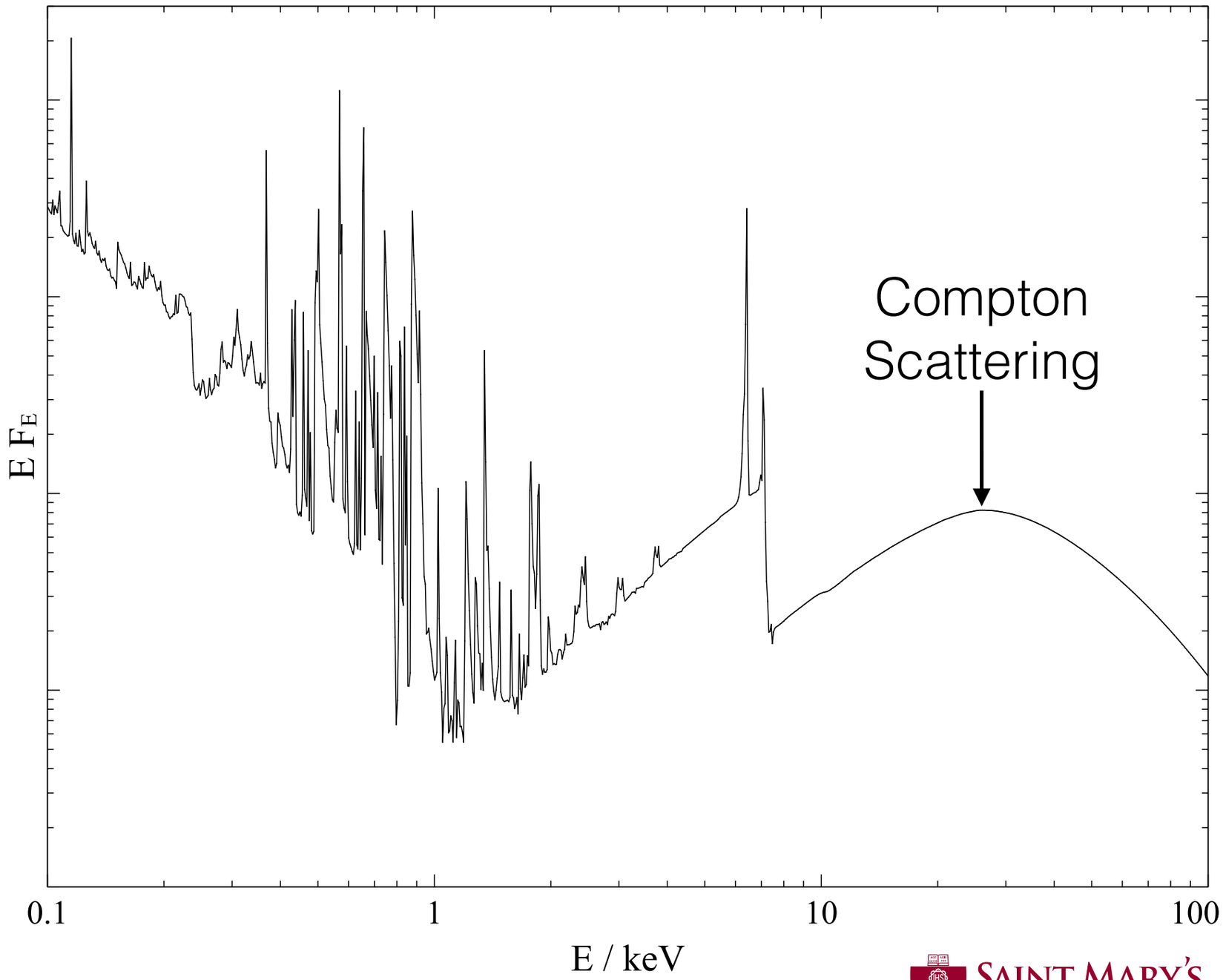
Reflection from Accretion Discs



Irradiate disc with power law







Ross & Fabian 2005, García et al. 2010

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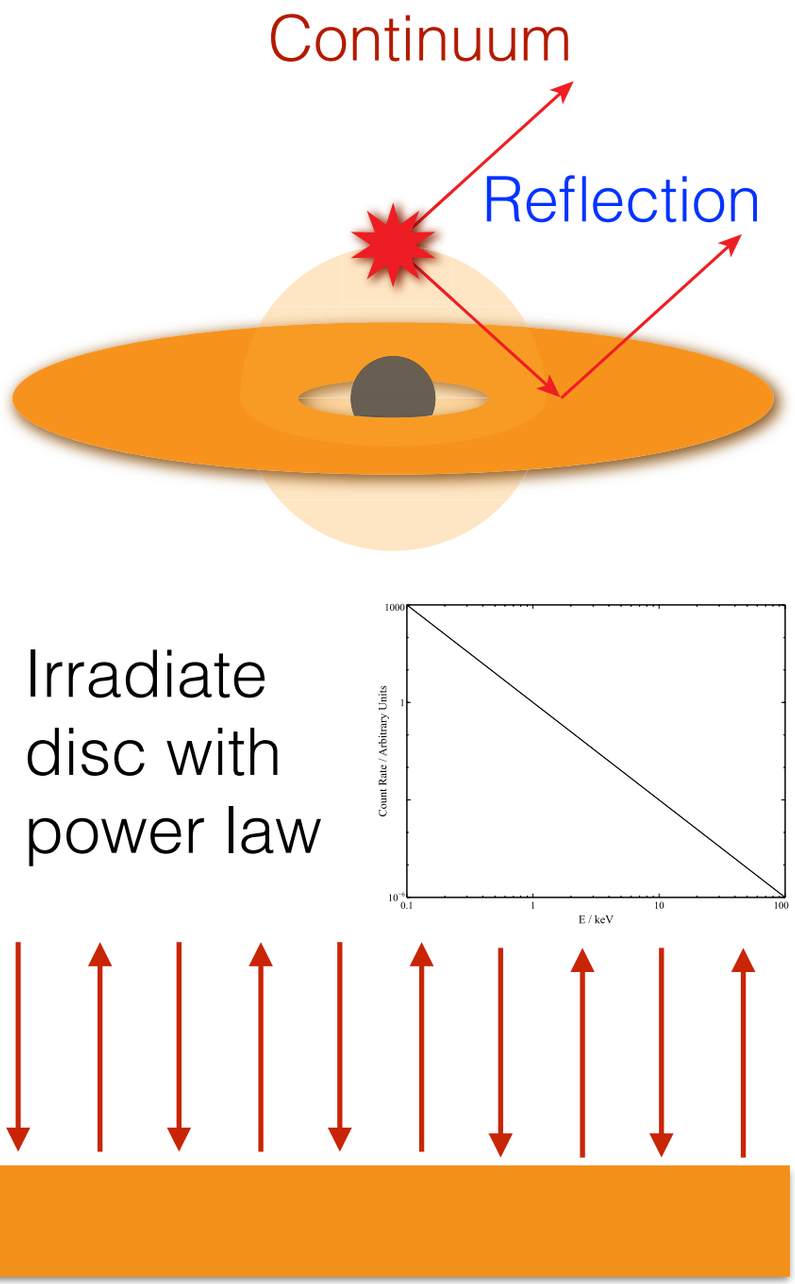
Reflection from Accretion Discs



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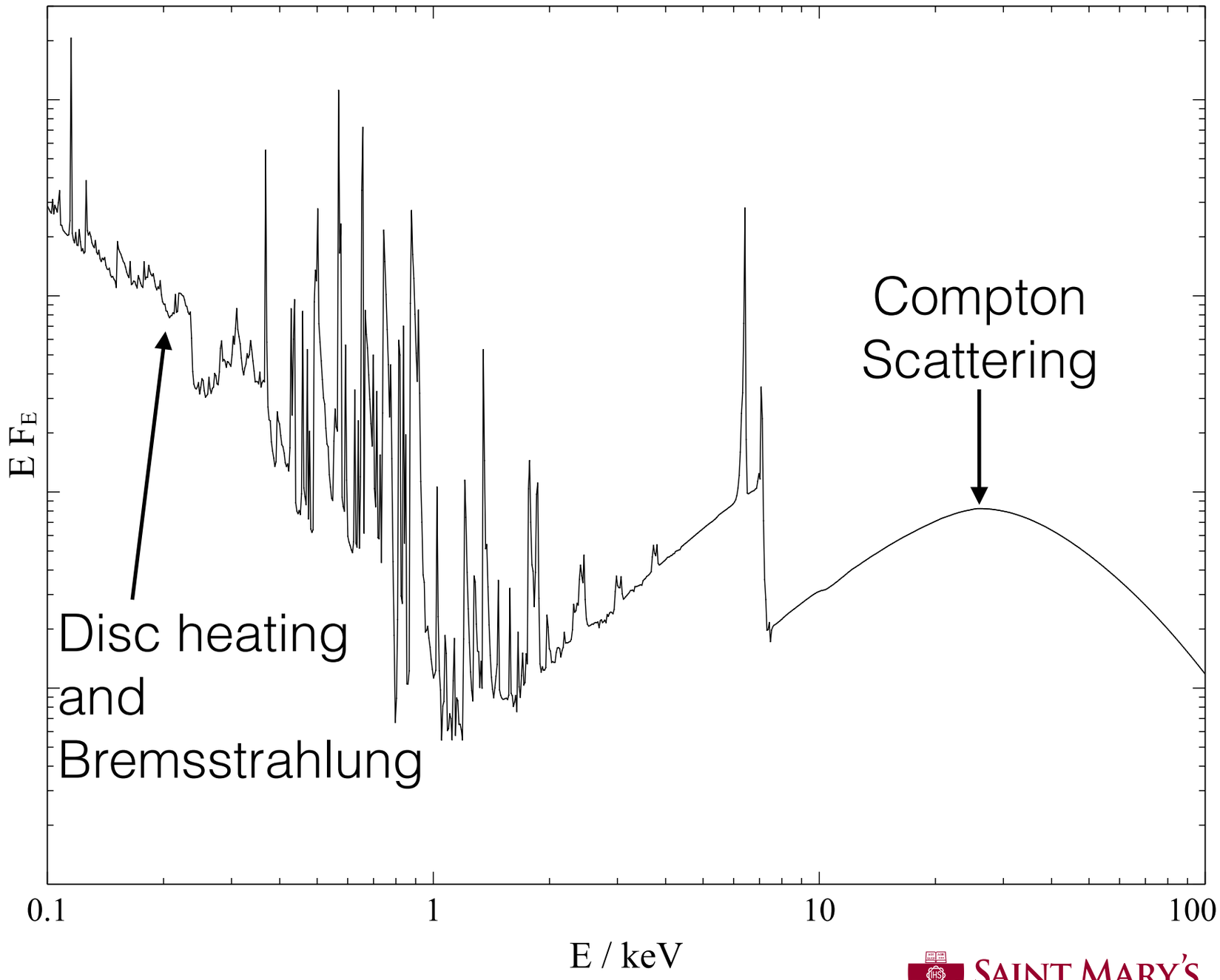

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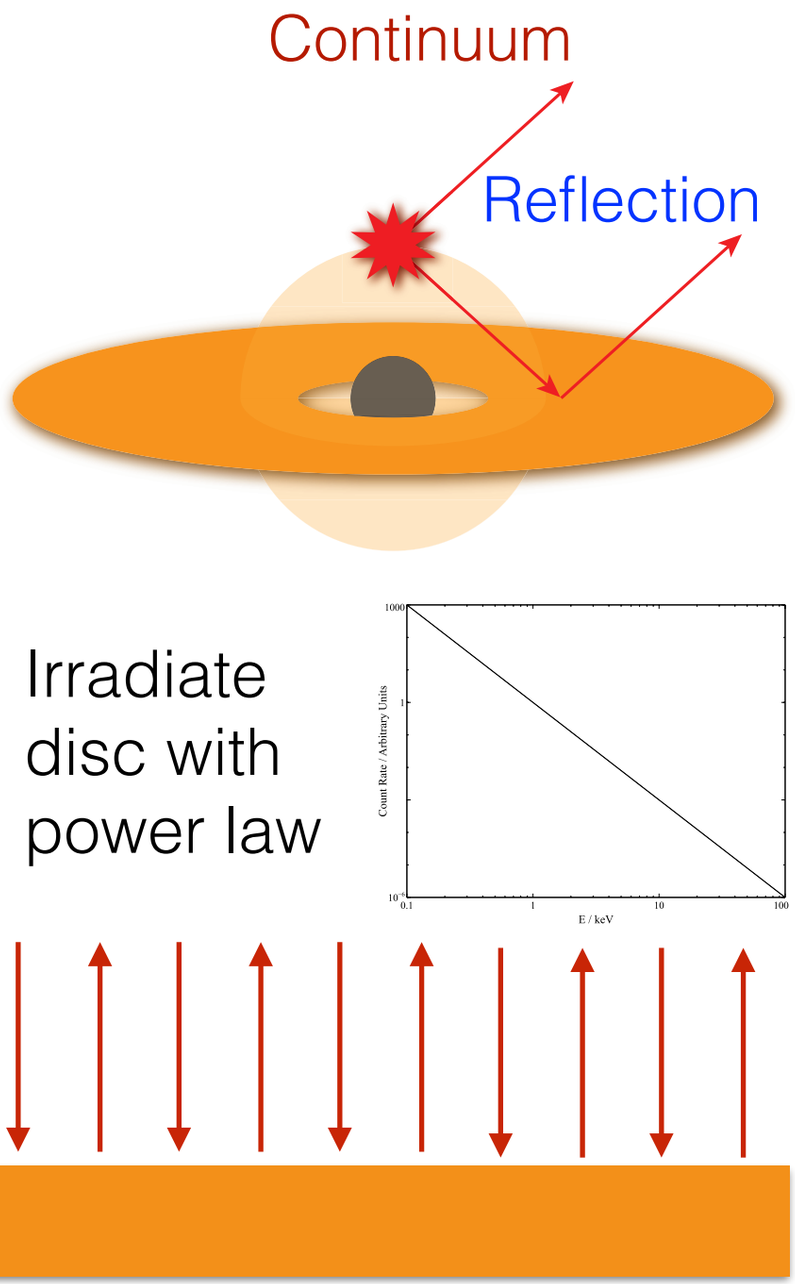

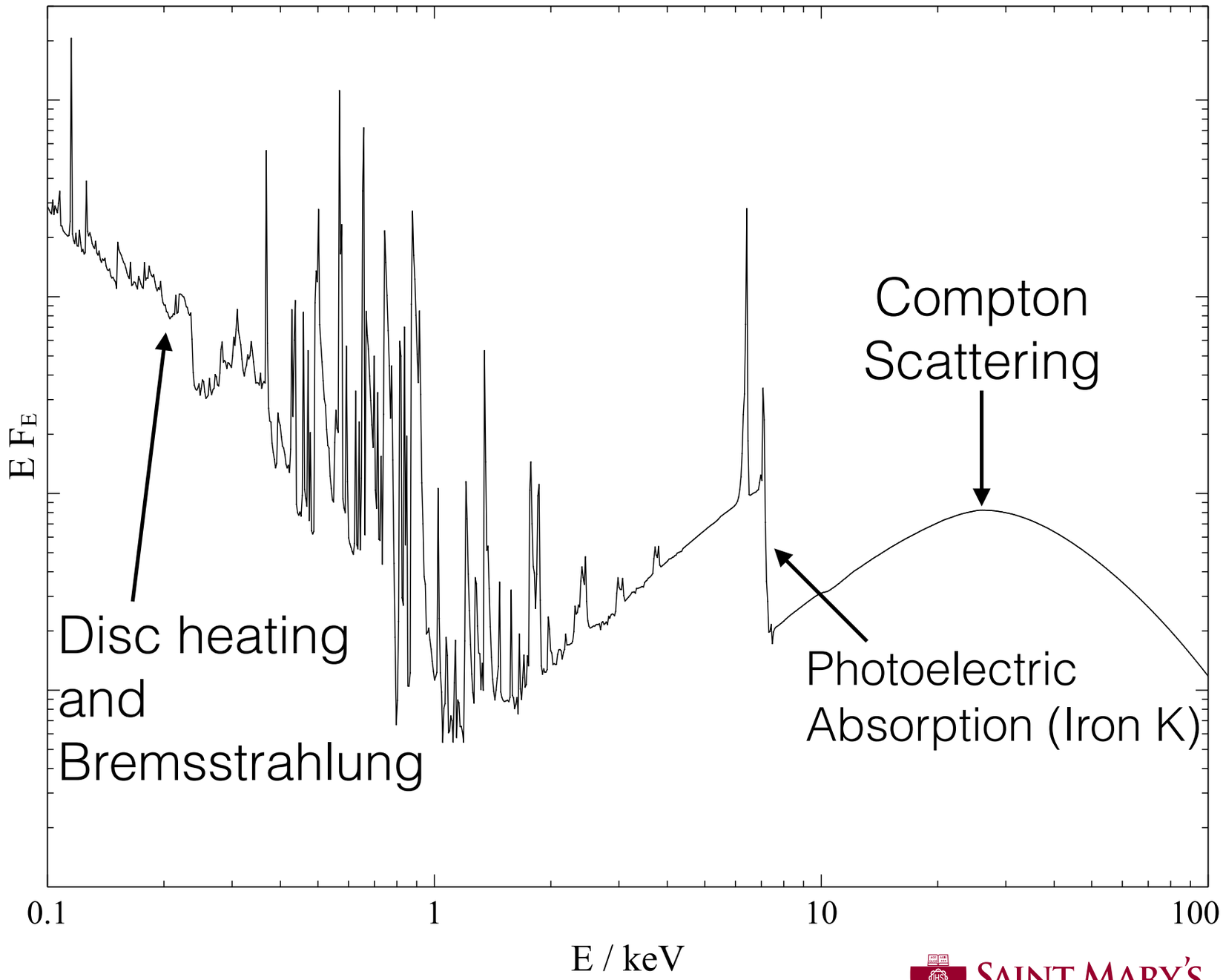

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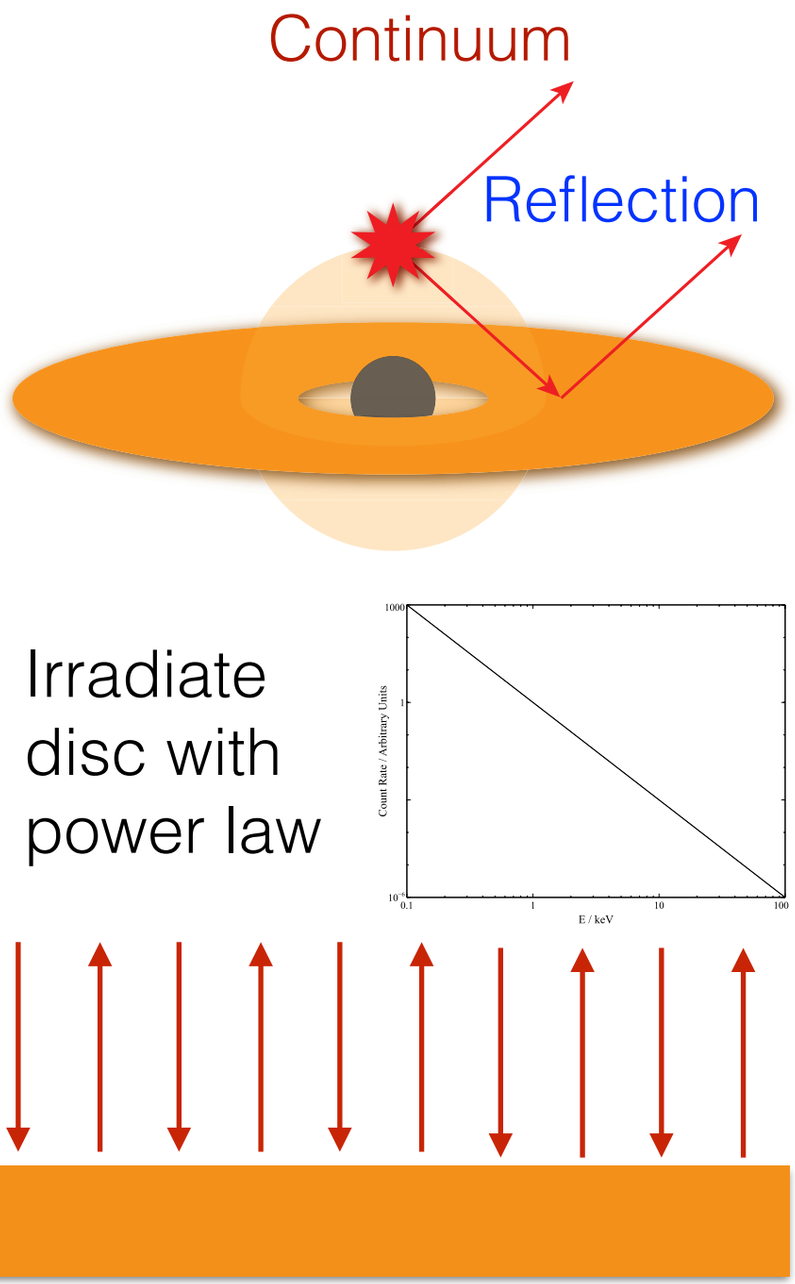

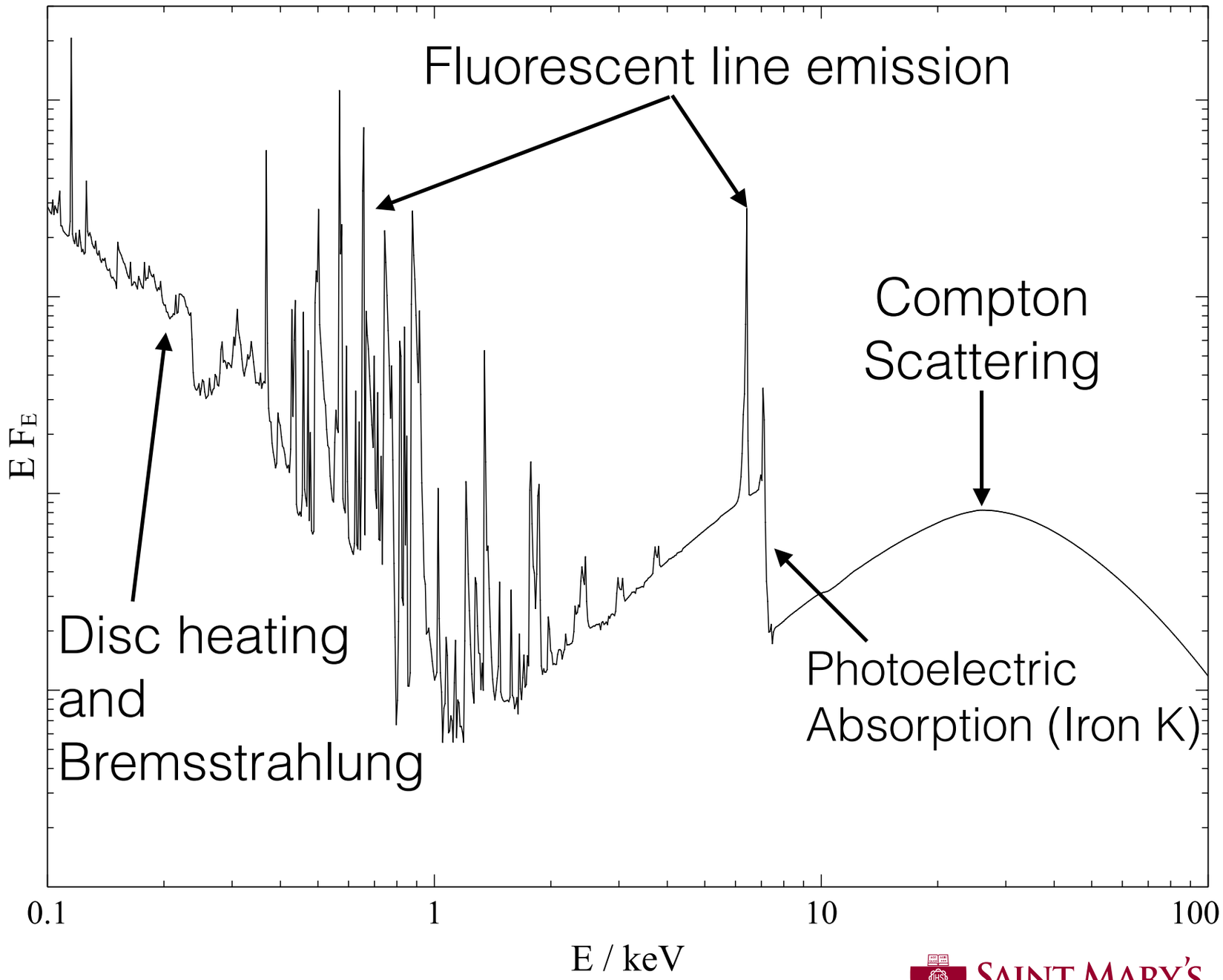

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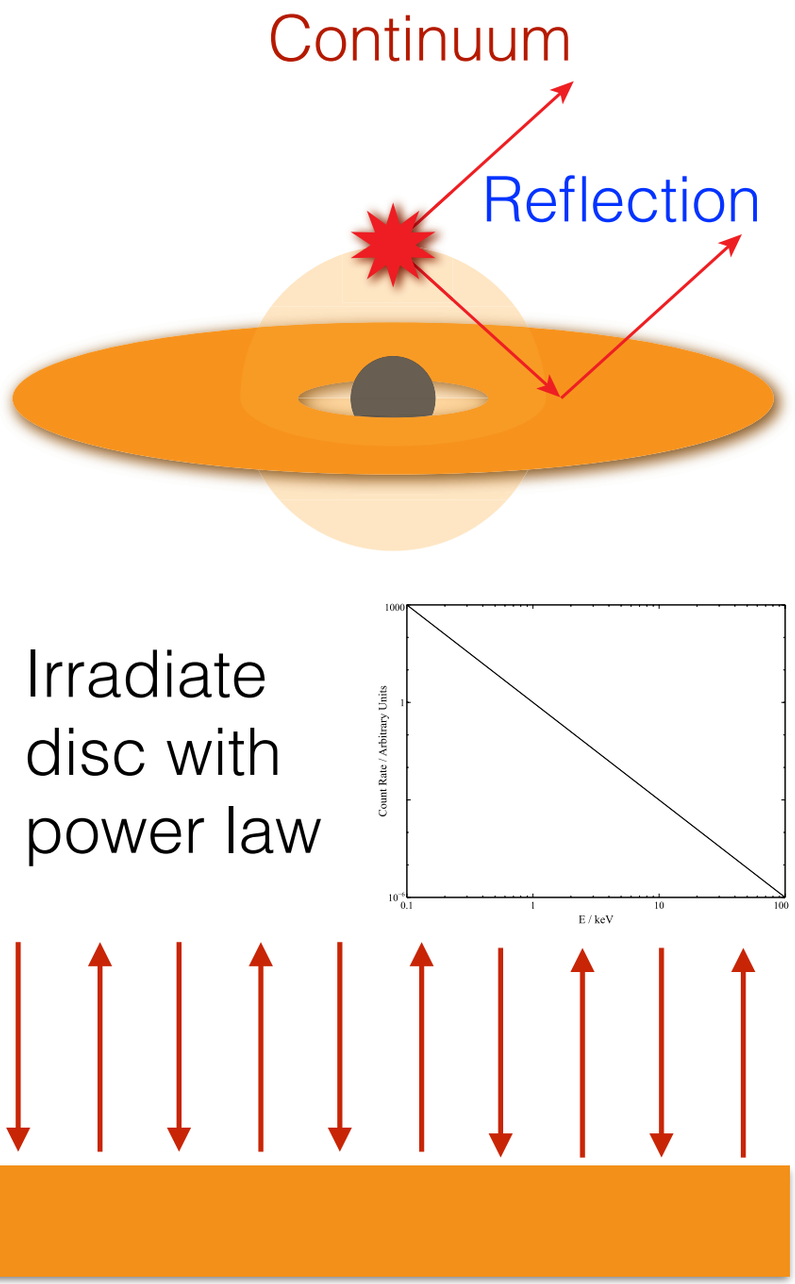

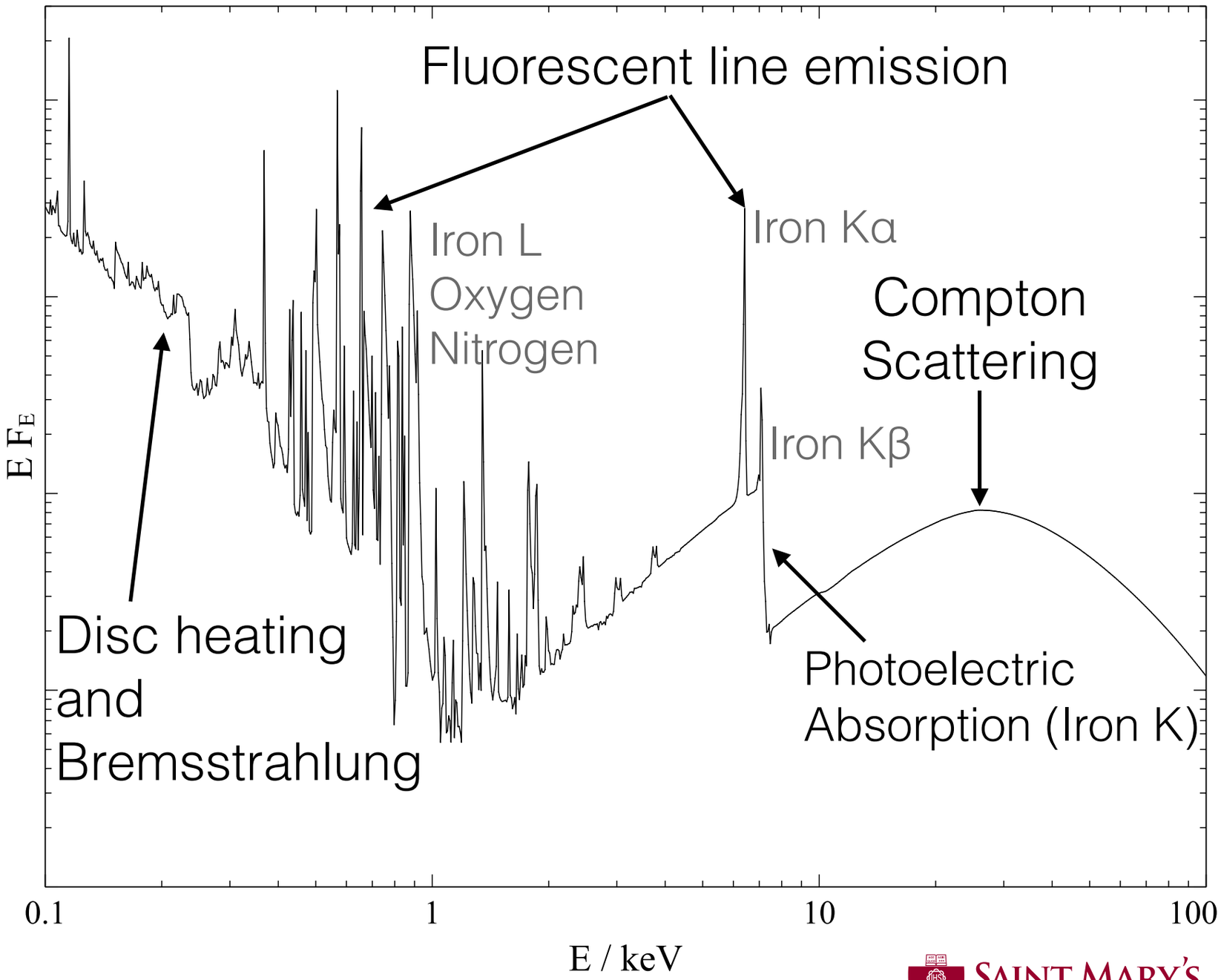

Ross & Fabian 2005, García et al. 2010

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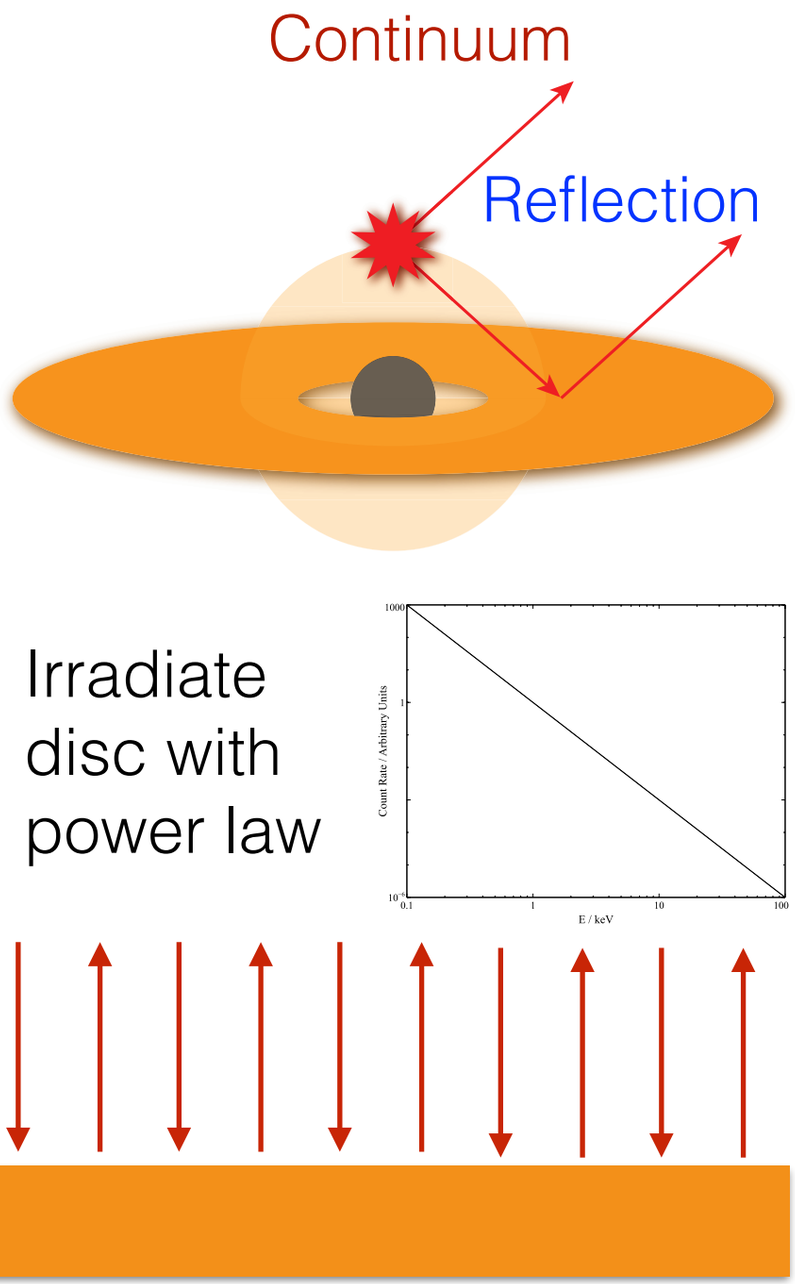
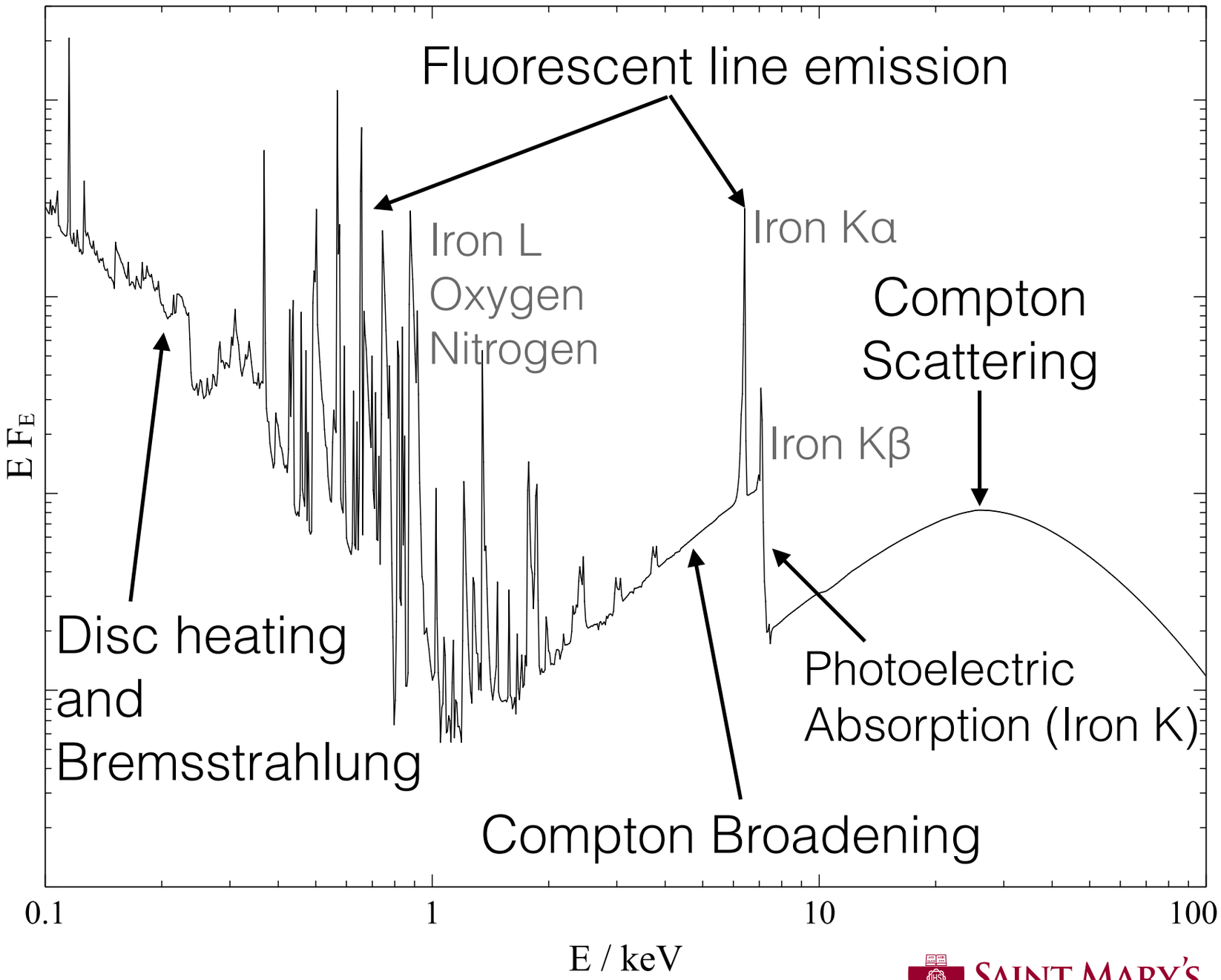
Reflection from Accretion Discs





Ross & Fabian 2005, García et al. 2010

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What would an emission line look like on the spectrum reflected from a black hole accretion disc?

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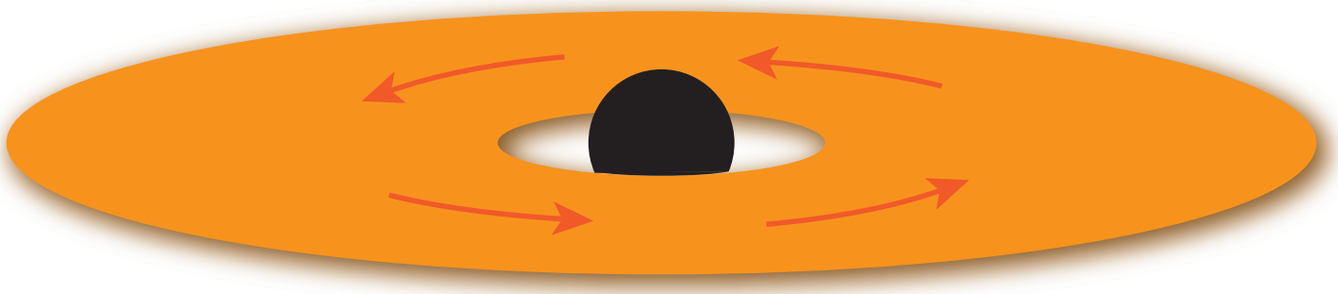
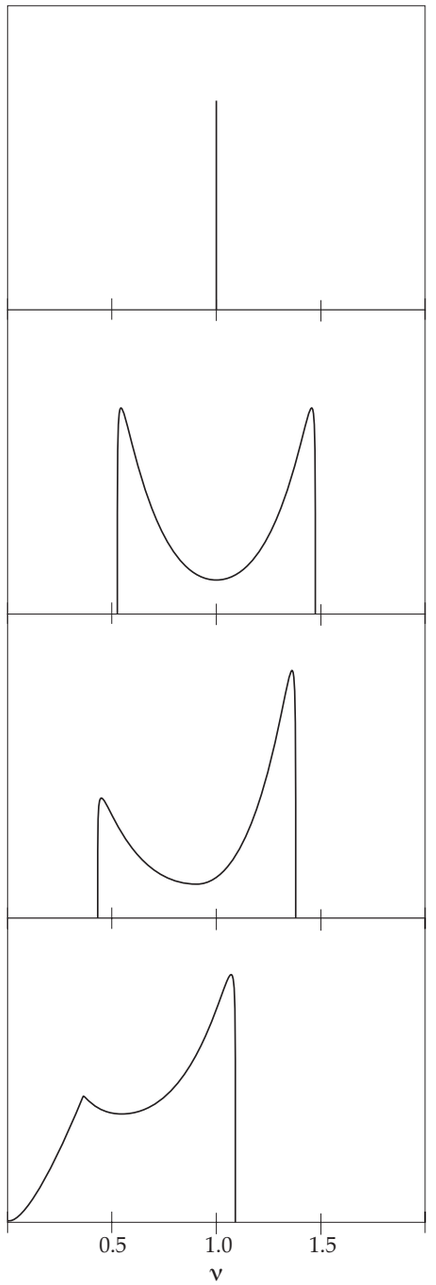
Relativistic Emission Lines





Fabian et al. 1989, Laor 1991

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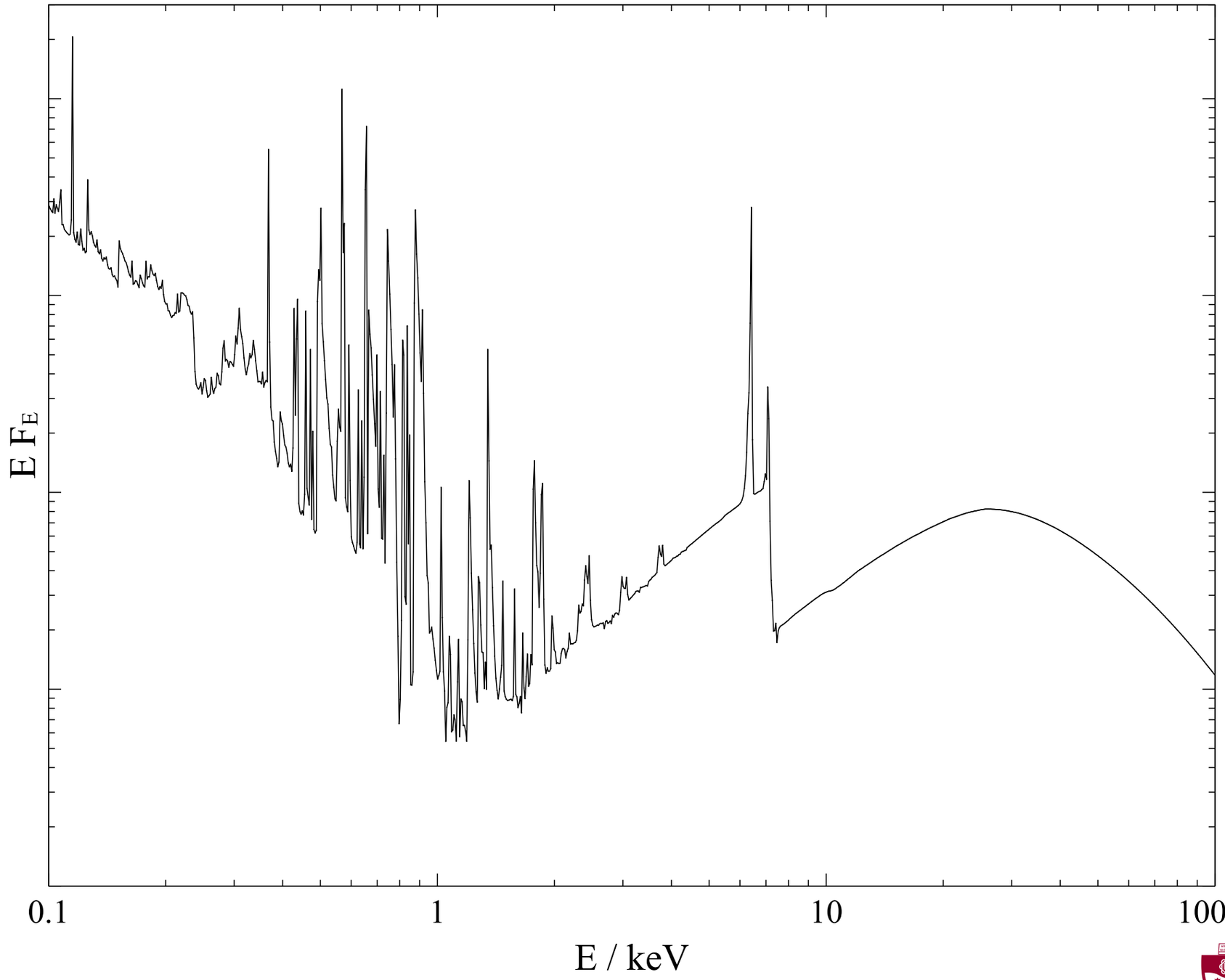
Blurred Reflection



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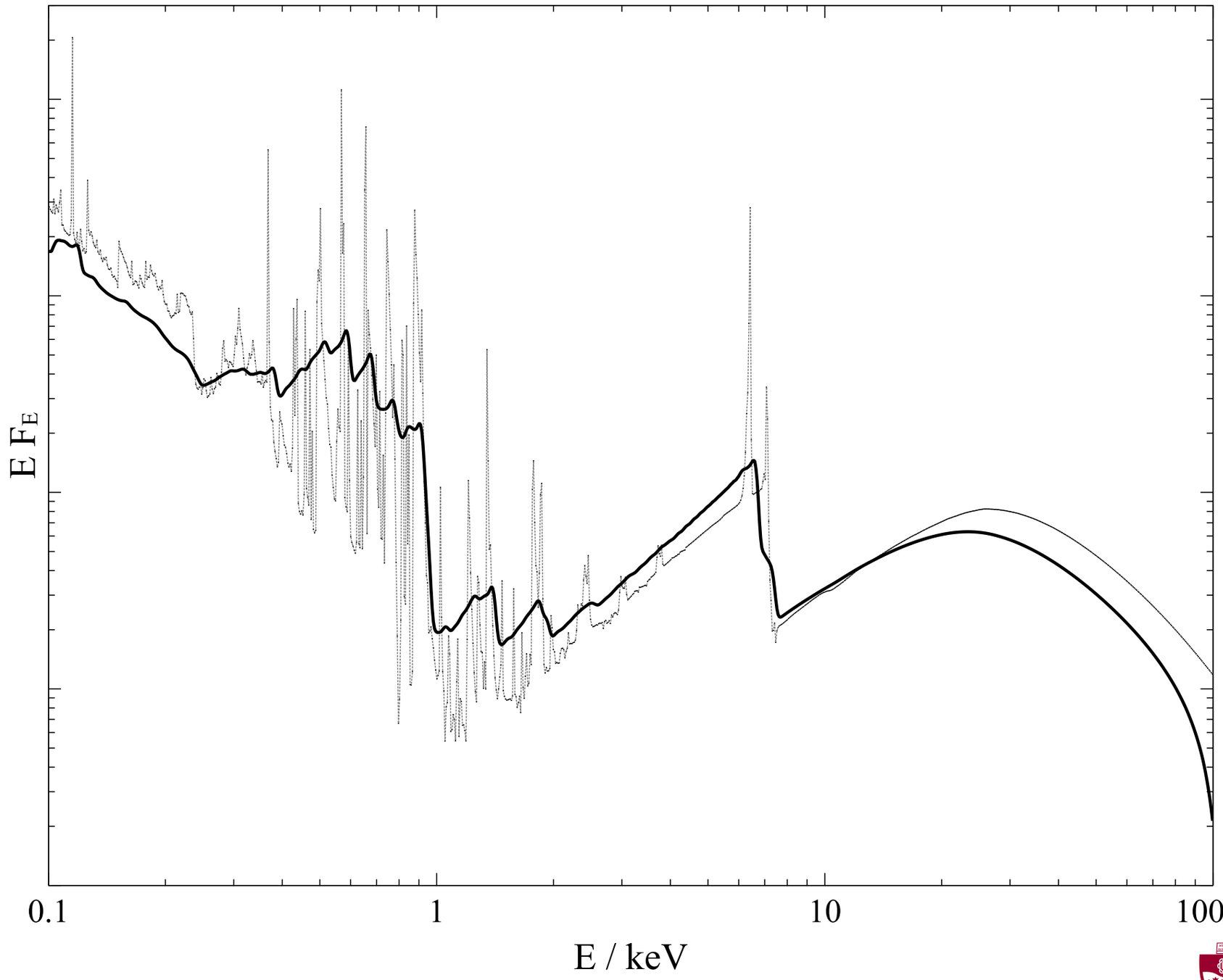
Blurred Reflection



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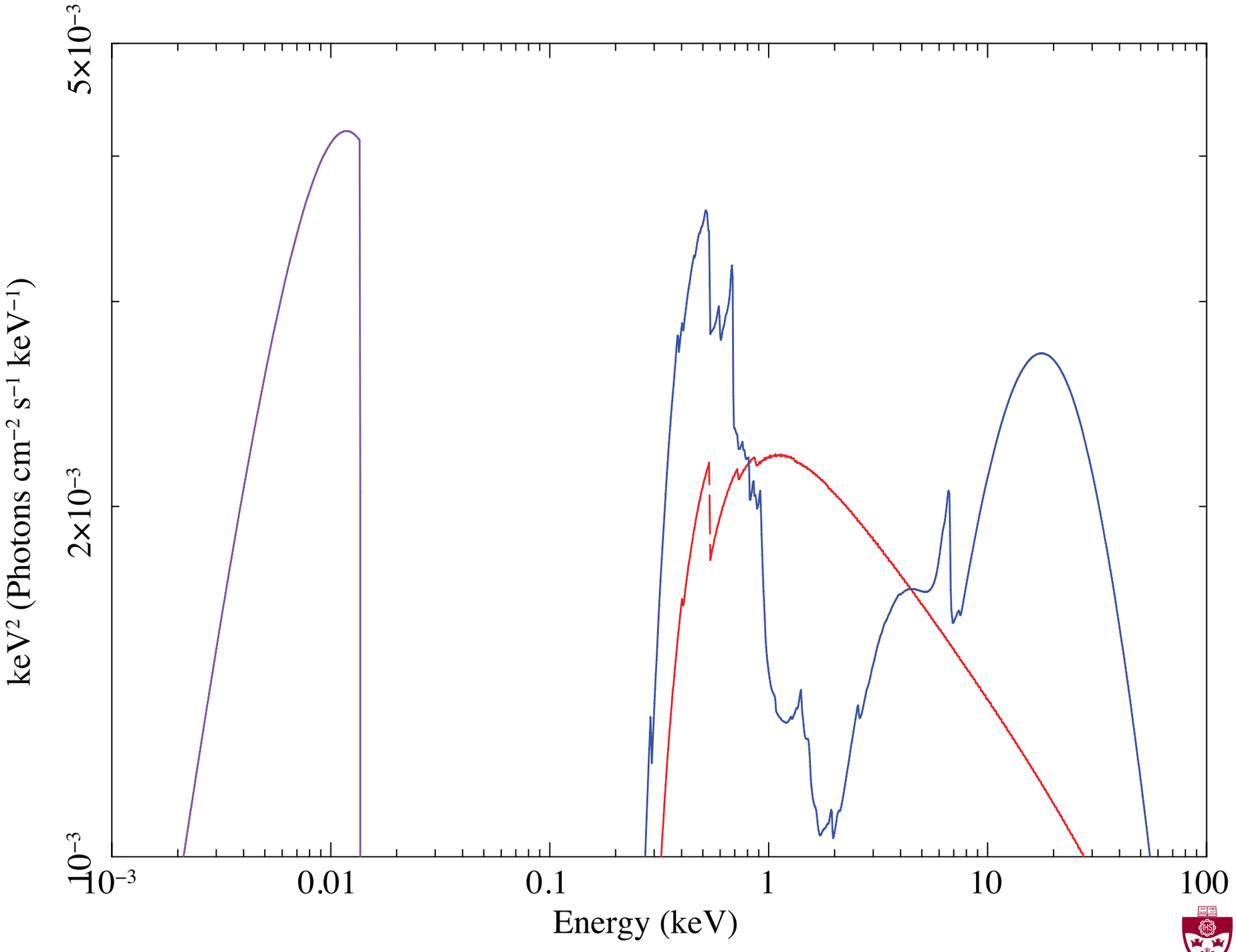
Putting the Spectrum Together



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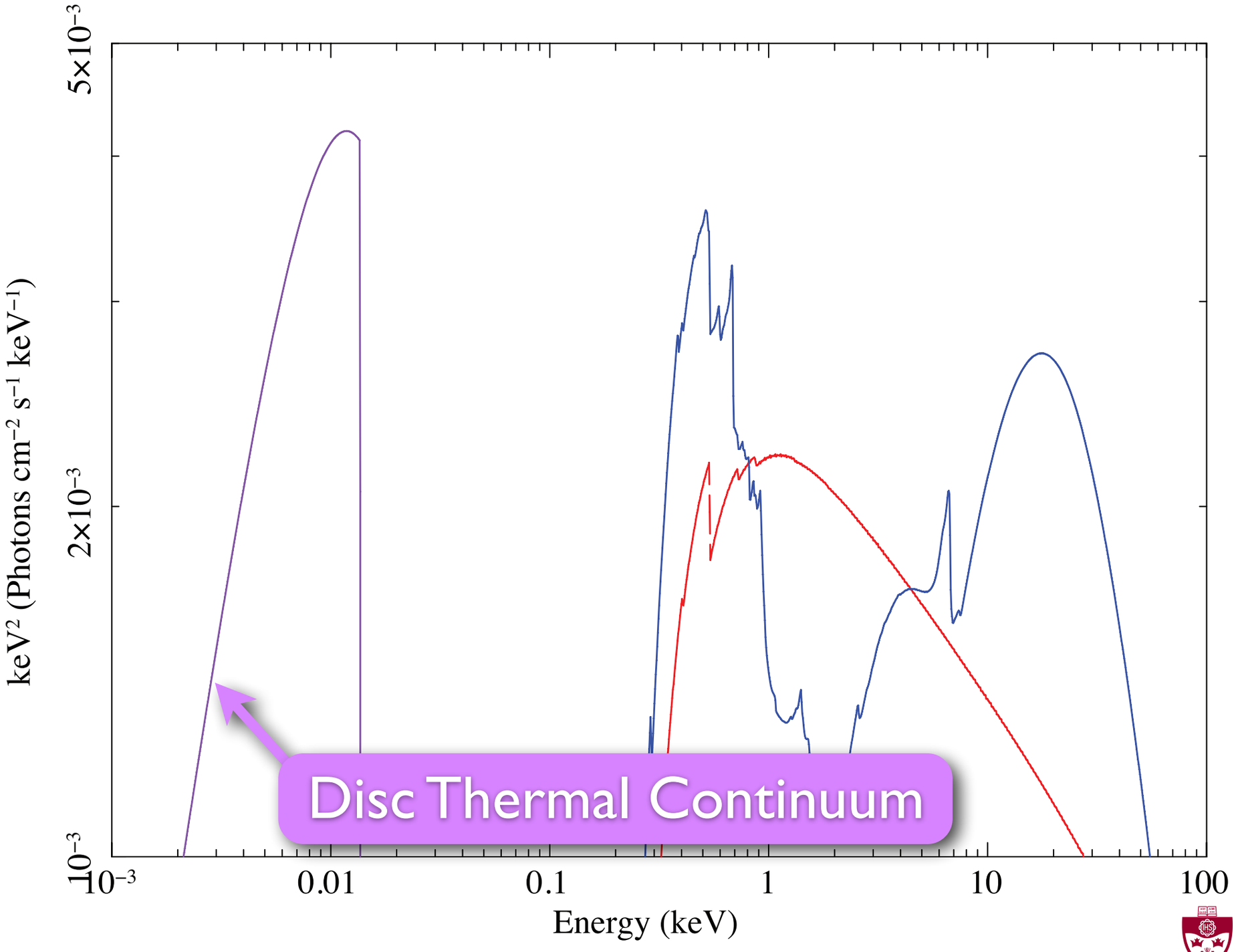

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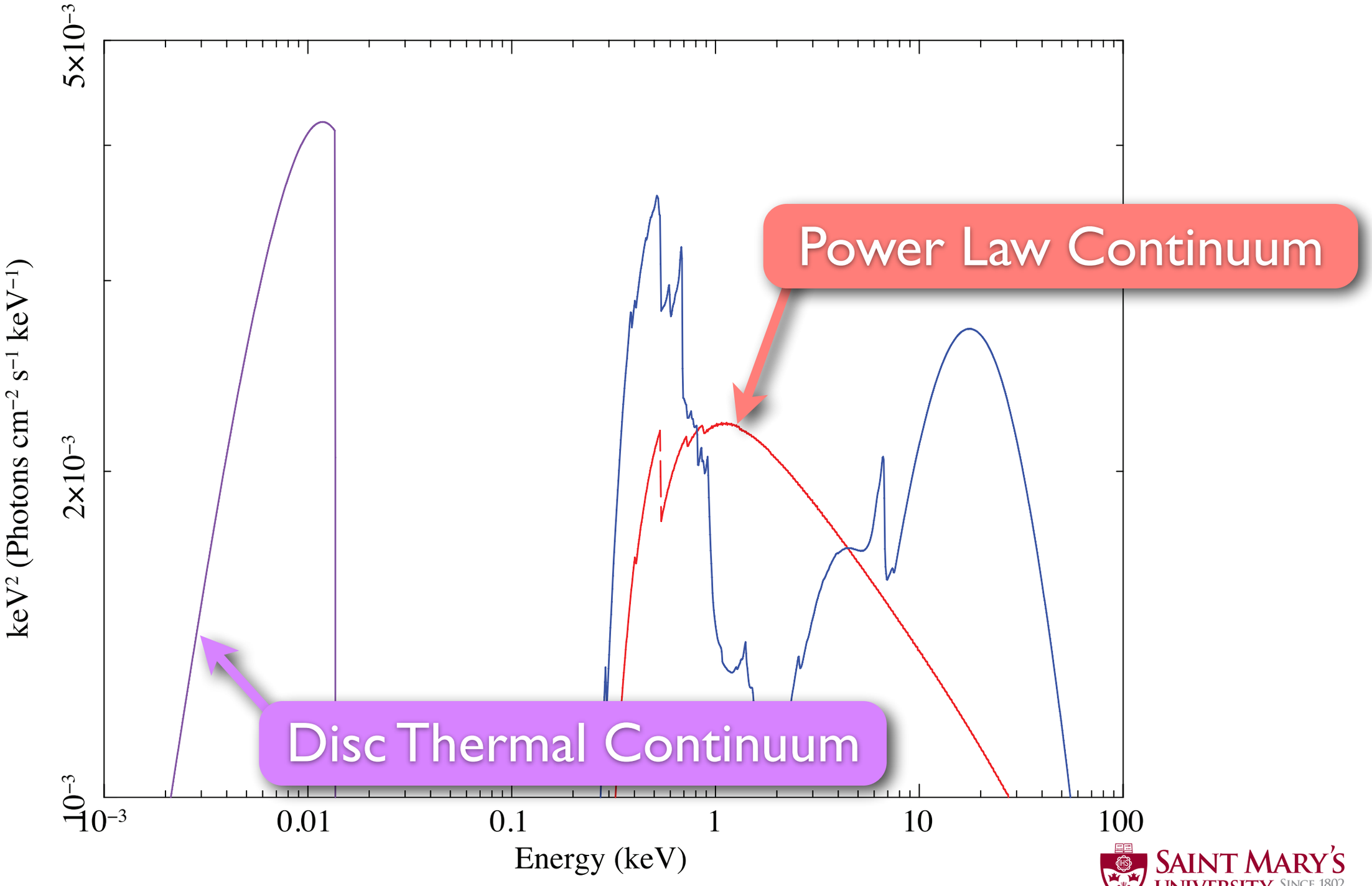

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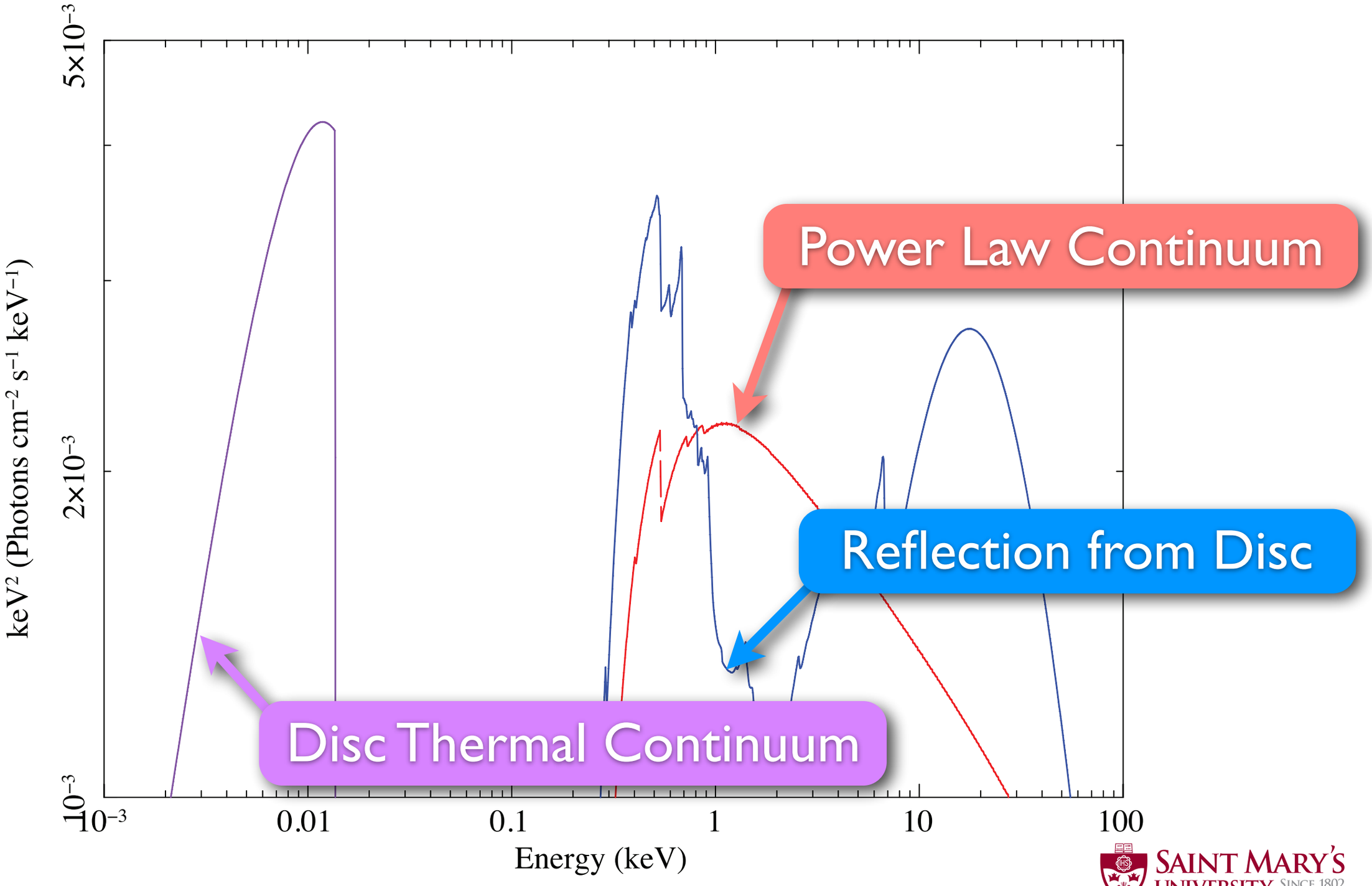

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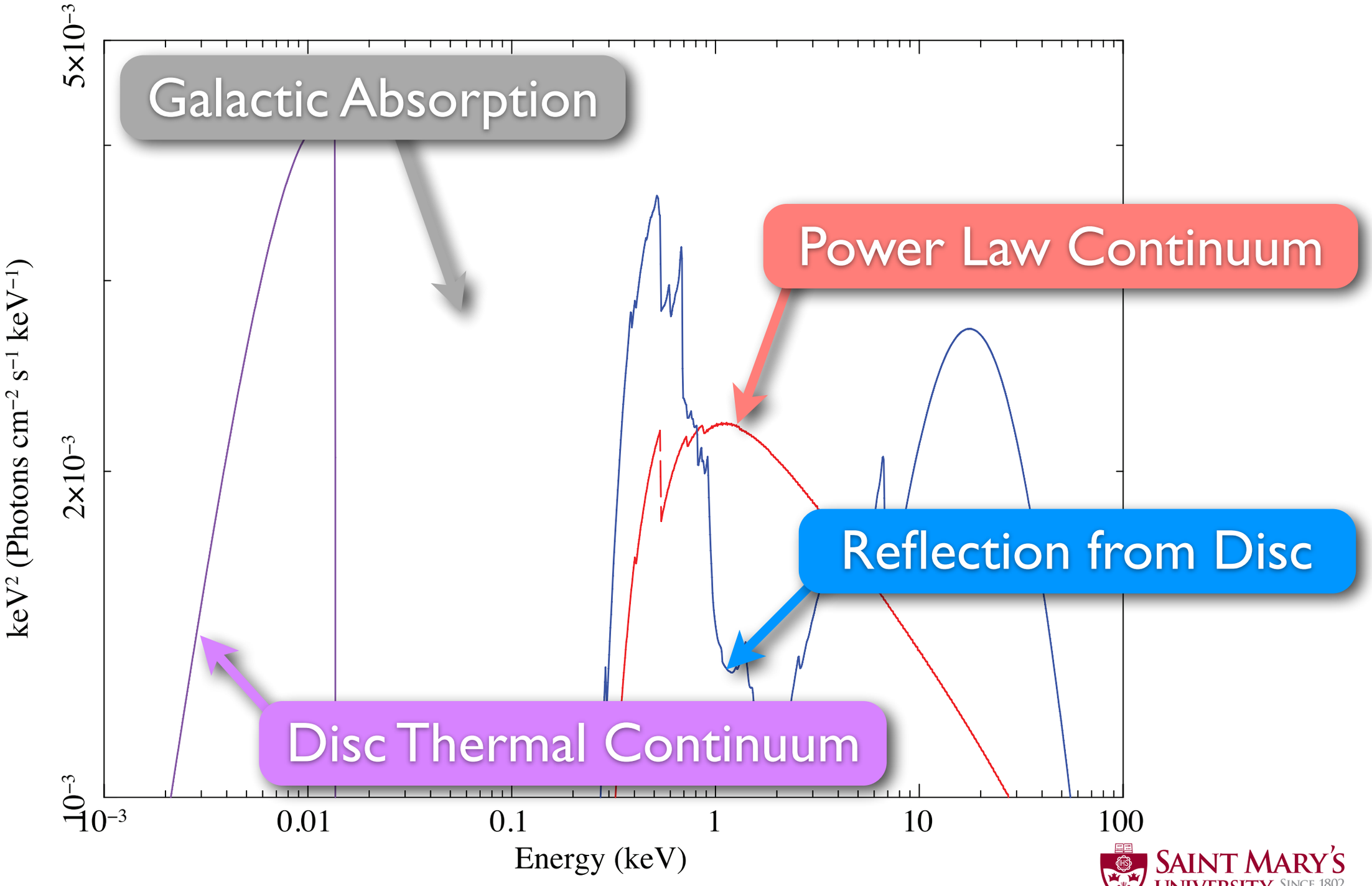

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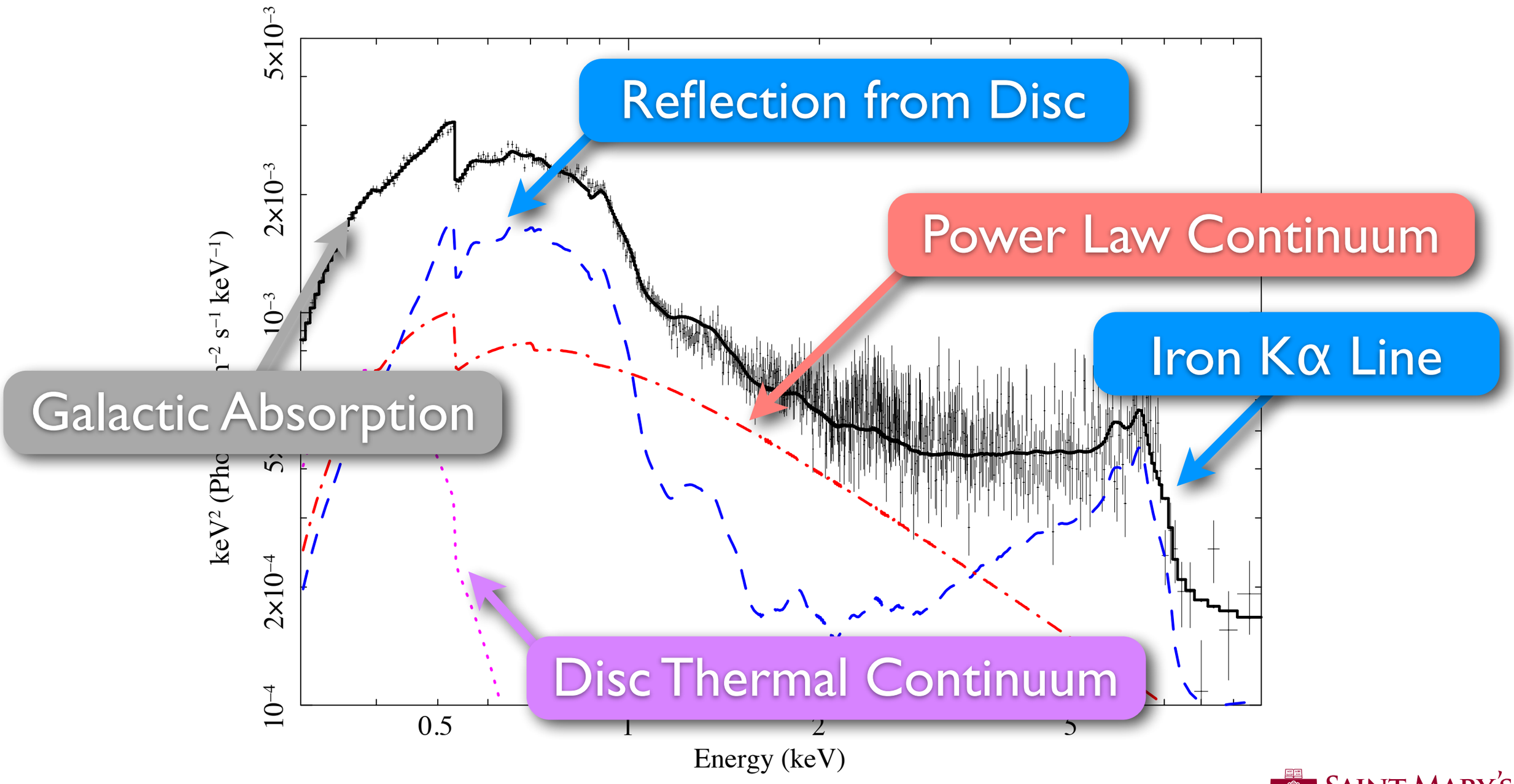
The X-ray Spectrum



Narrow line Seyfert 1 (AGN), 1H 0707–495 (Fabian et al. 2009)

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Ray Tracing

Photons propagate along null geodesics

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0 \qquad\qquad g_{ab}\dot{x}^a\dot{x}^b = 0$$

First order differential equations (integrate numerically)

$$\dot{t} = \frac{\left[(r^2 + a^2\cos^2\theta)(r^2 + a^2) + 2a^2 r\sin^2\theta\right]k - 2arh}{r^2\left(1 + \frac{a^2\cos^2\theta}{r^2} - \frac{2}{r}\right)(r^2 + a^2) + 2a^2 r\sin^2\theta}$$

$$\dot{\varphi} = \frac{2ark\sin^2\theta + (r^2 + a^2\cos^2\theta - 2r)h}{(r^2 + a^2)(r^2 + a^2\cos^2\theta - 2r)\sin^2\theta + 2a^2 r\sin^4\theta}$$

$$\dot{\theta}^2 = \frac{Q + (ka\cos\theta - h\cot\theta)(ka\cos\theta + h\cot\theta)}{\rho^4}$$

$$\dot{r}^2 = \frac{\Delta}{\rho^2}\left[k\dot{t} - h\dot{\varphi} - \rho^2\dot{\theta}^2\right]$$

And the redshift $$\frac{E_R}{E_E} = \frac{\mathbf{u}_R \cdot \mathbf{p}(R)}{\mathbf{u}_E \cdot \mathbf{p}(E)}$$

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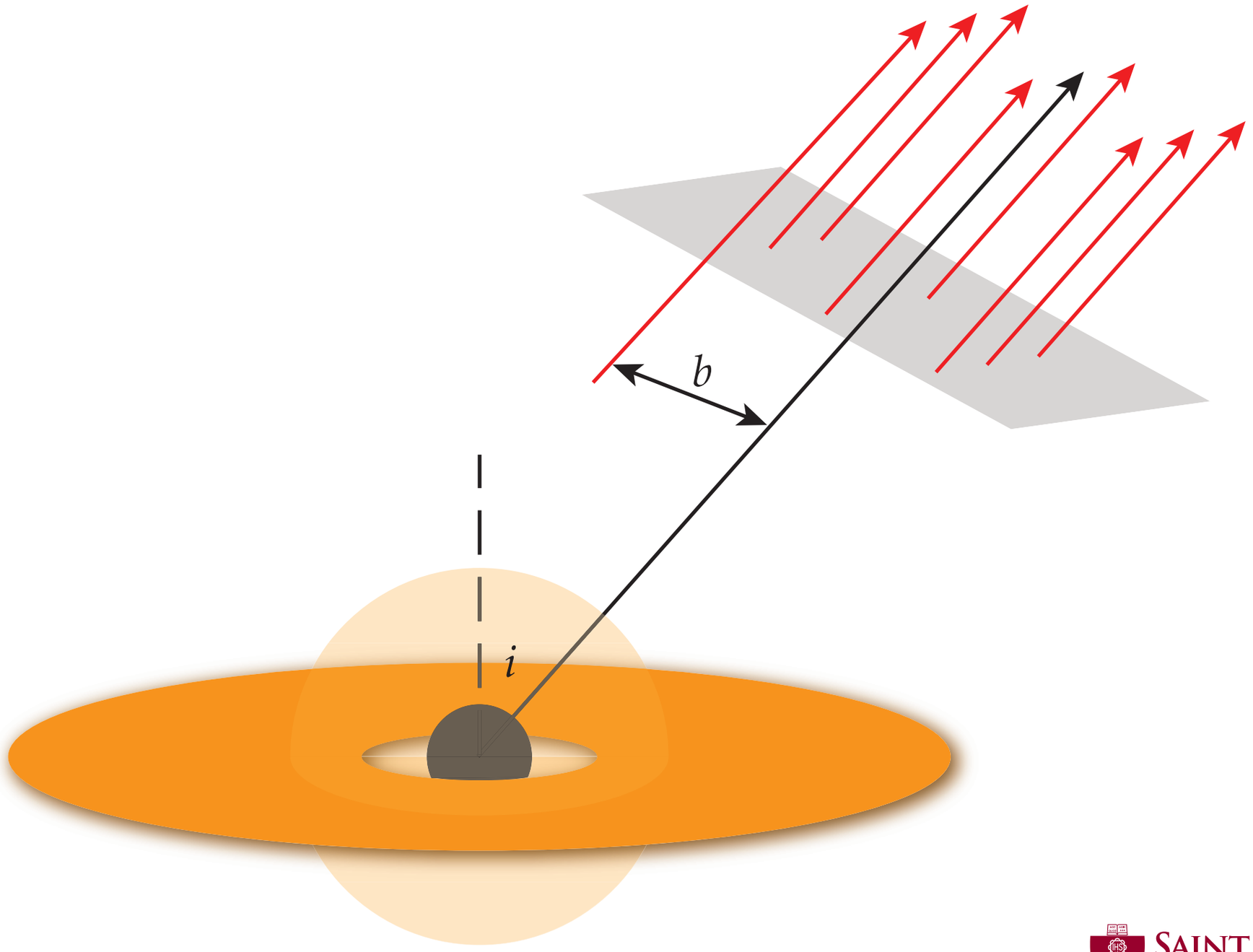
Observing the Accretion Disc



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What do we expect the disc to look like?

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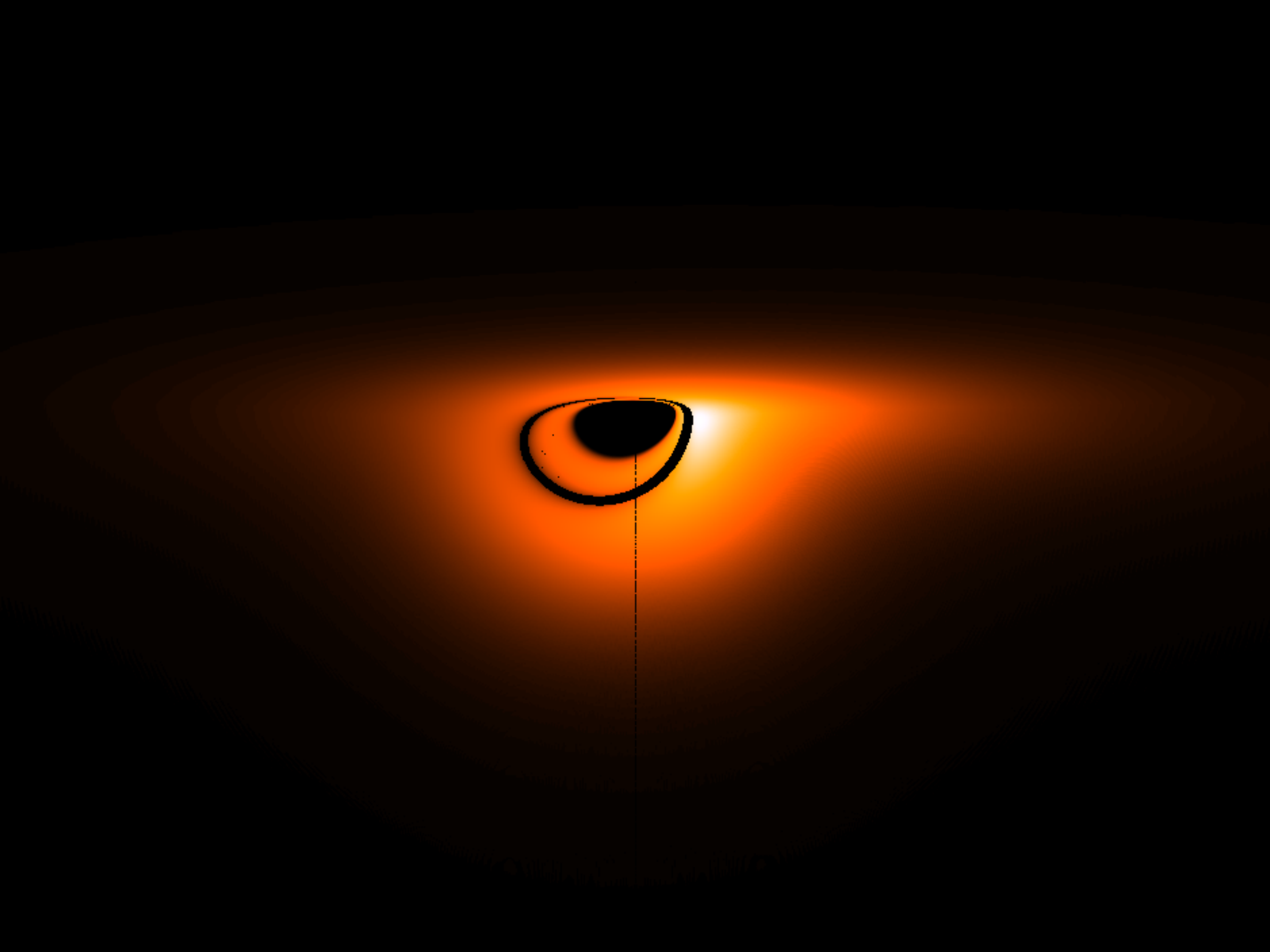

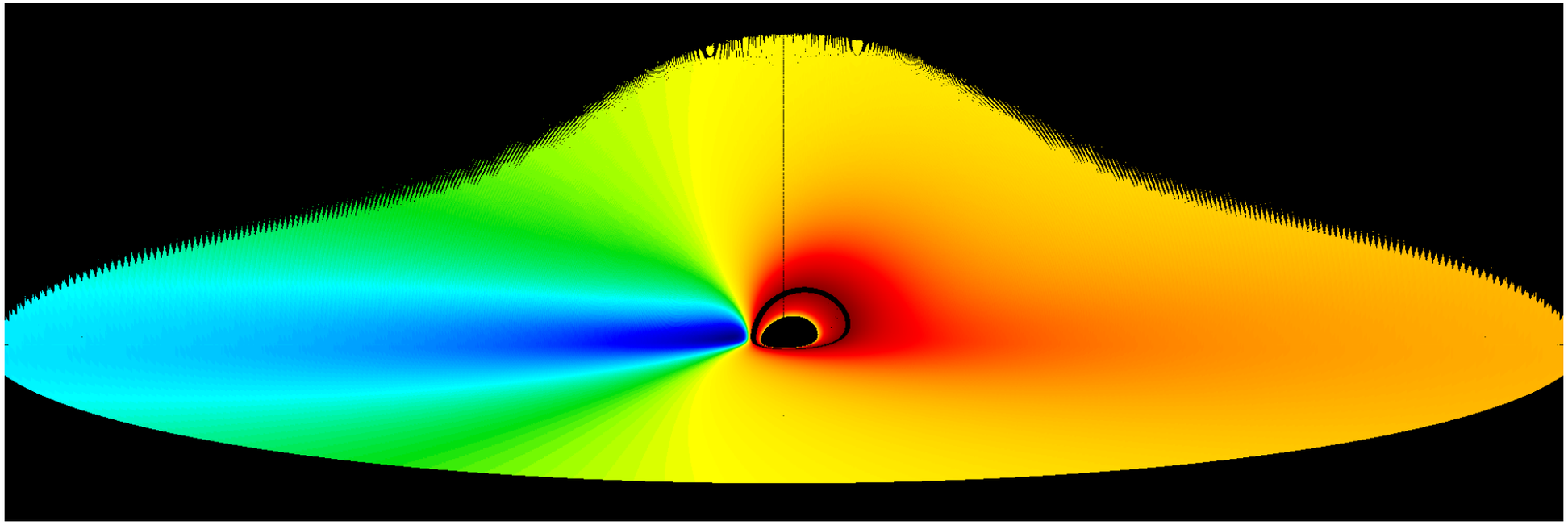

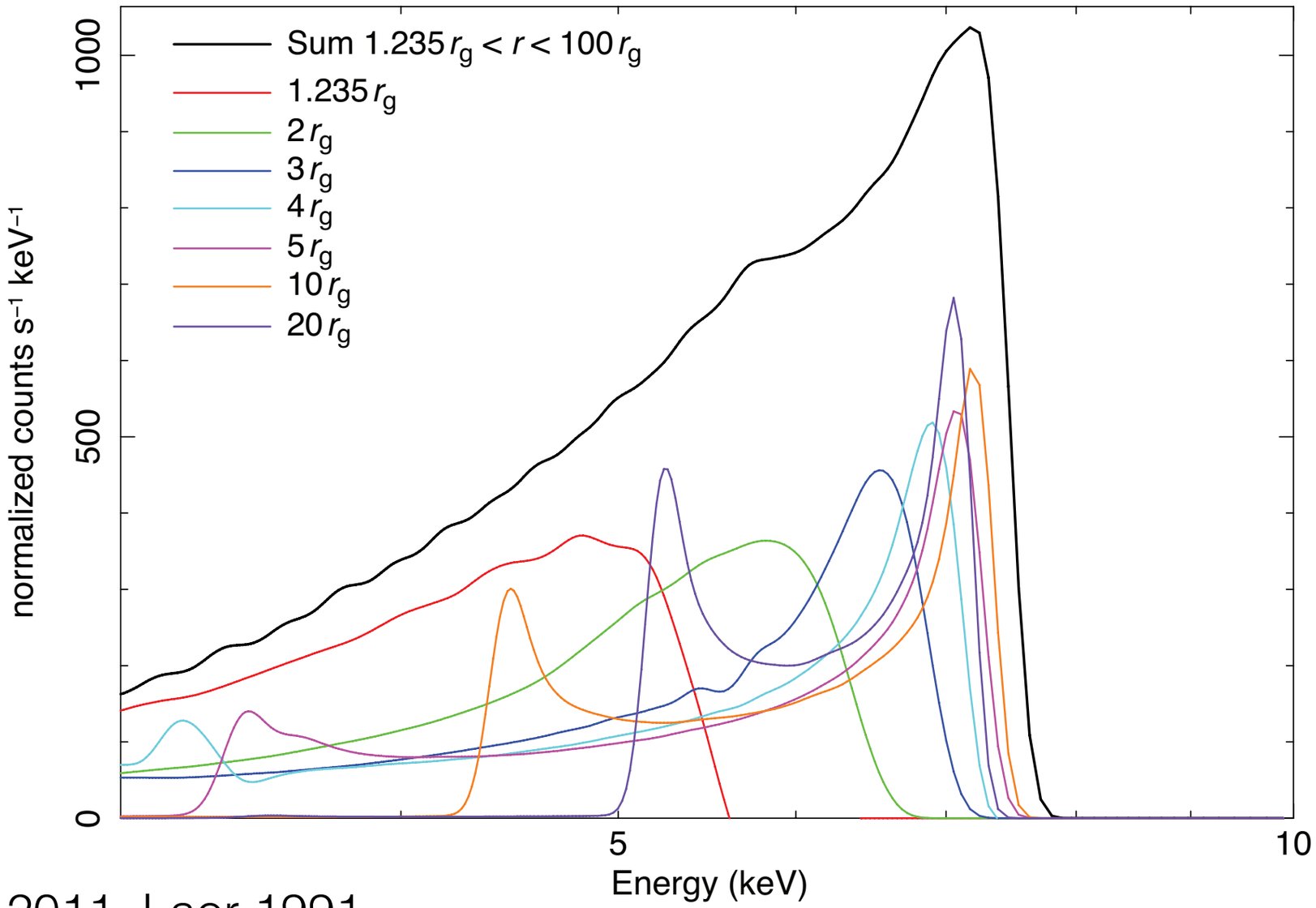

Wilkins & Fabian 2011, Laor 1991

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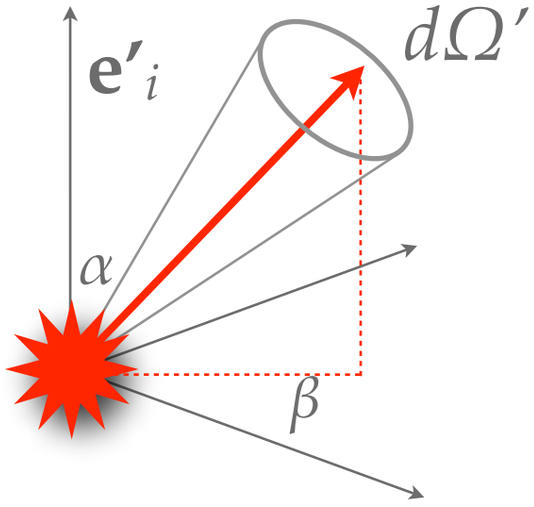
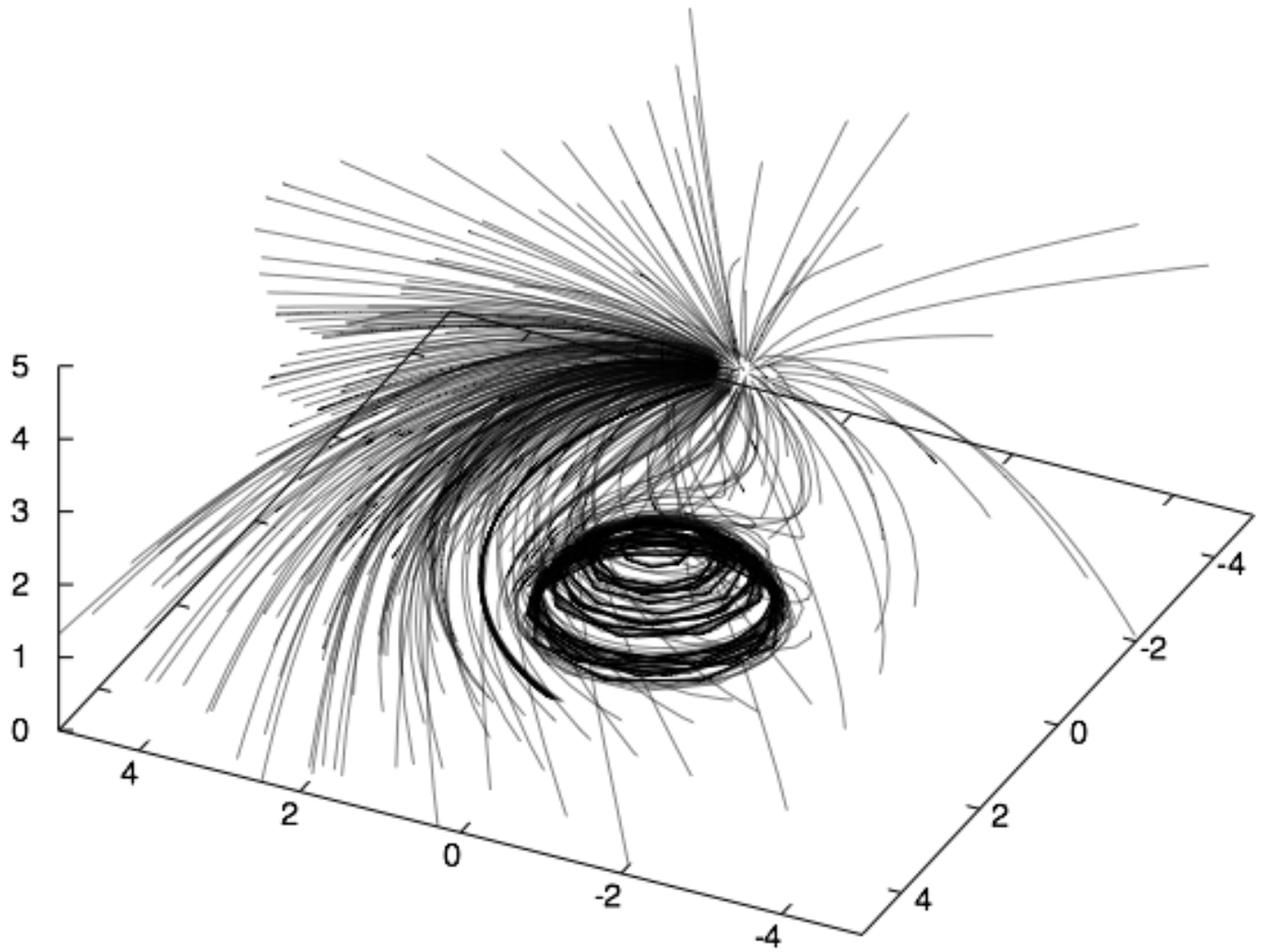
Illuminating the Disc





In the (locally flat) rest frame of the source

$$\mathbf{e}'_{(a)} \cdot \mathbf{e}'_{(b)} = \eta_{(a)(b)}$$

Start rays at equal interval in solid angle in source's frame

$$\mathbf{p}' = (E_0, E_0 \sin\alpha \cos\beta, E_0 \sin\alpha \sin\beta, E_0 \cos\alpha$$

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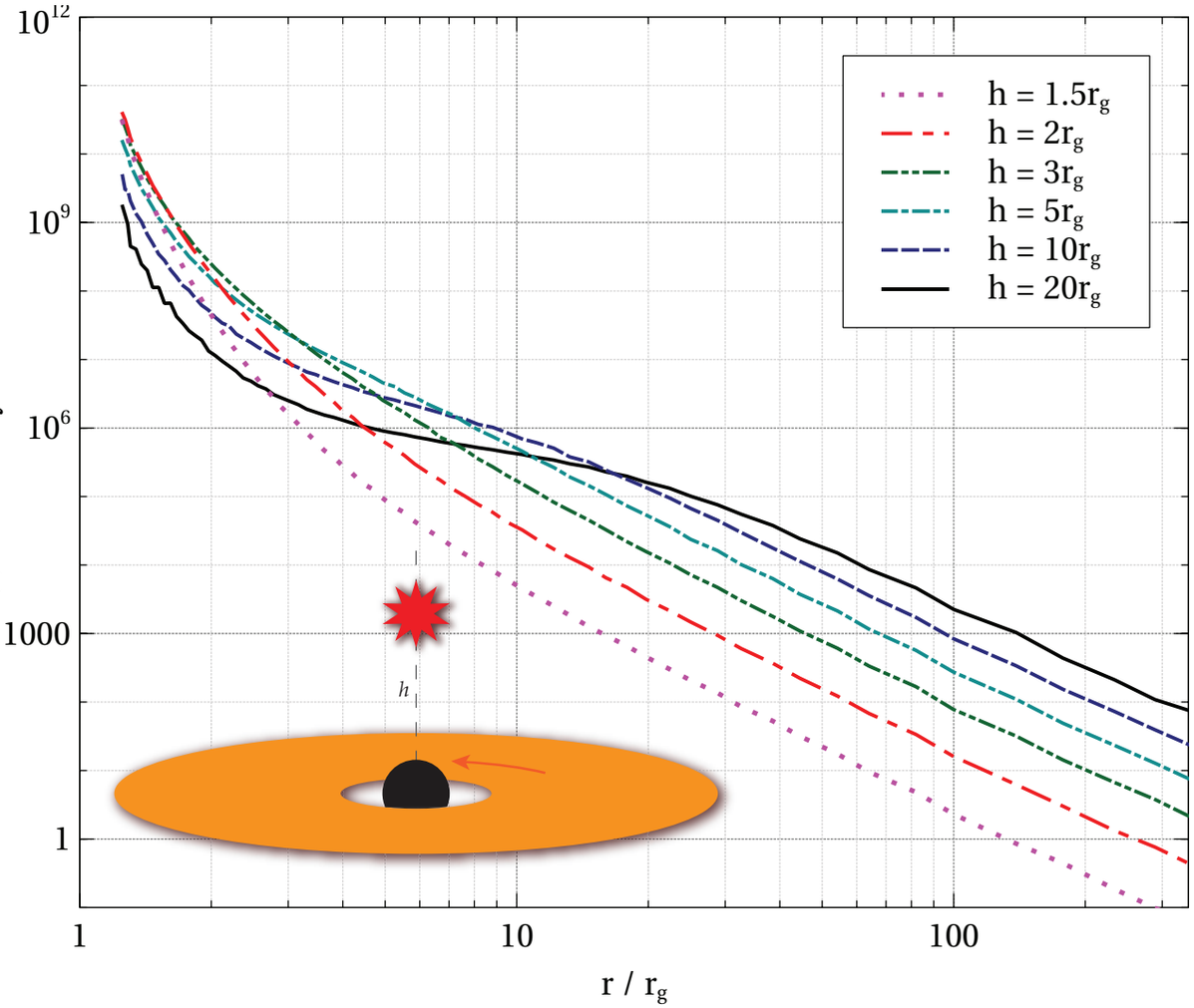
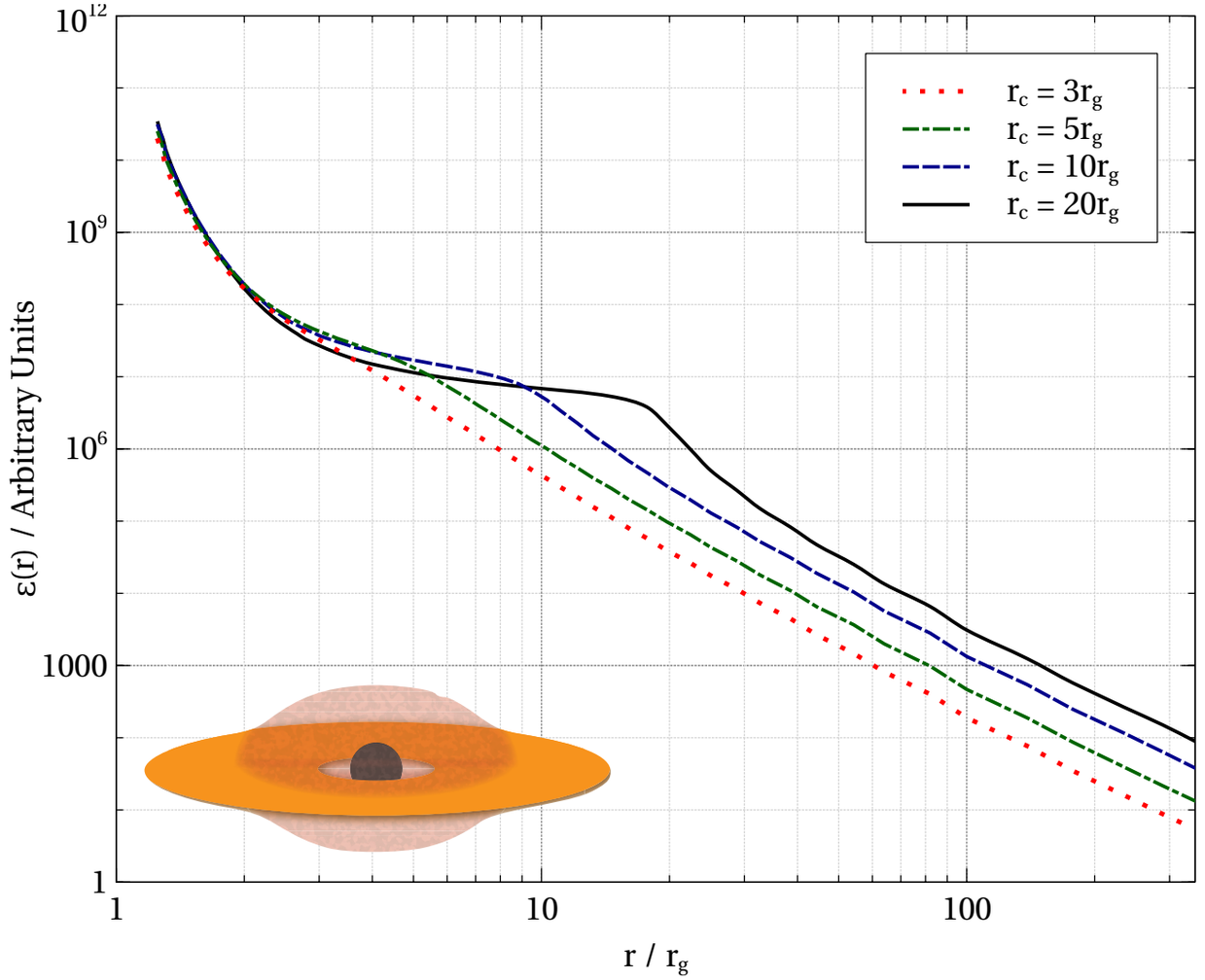
The Emissivity Profile





Wilkins & Fabian 2012

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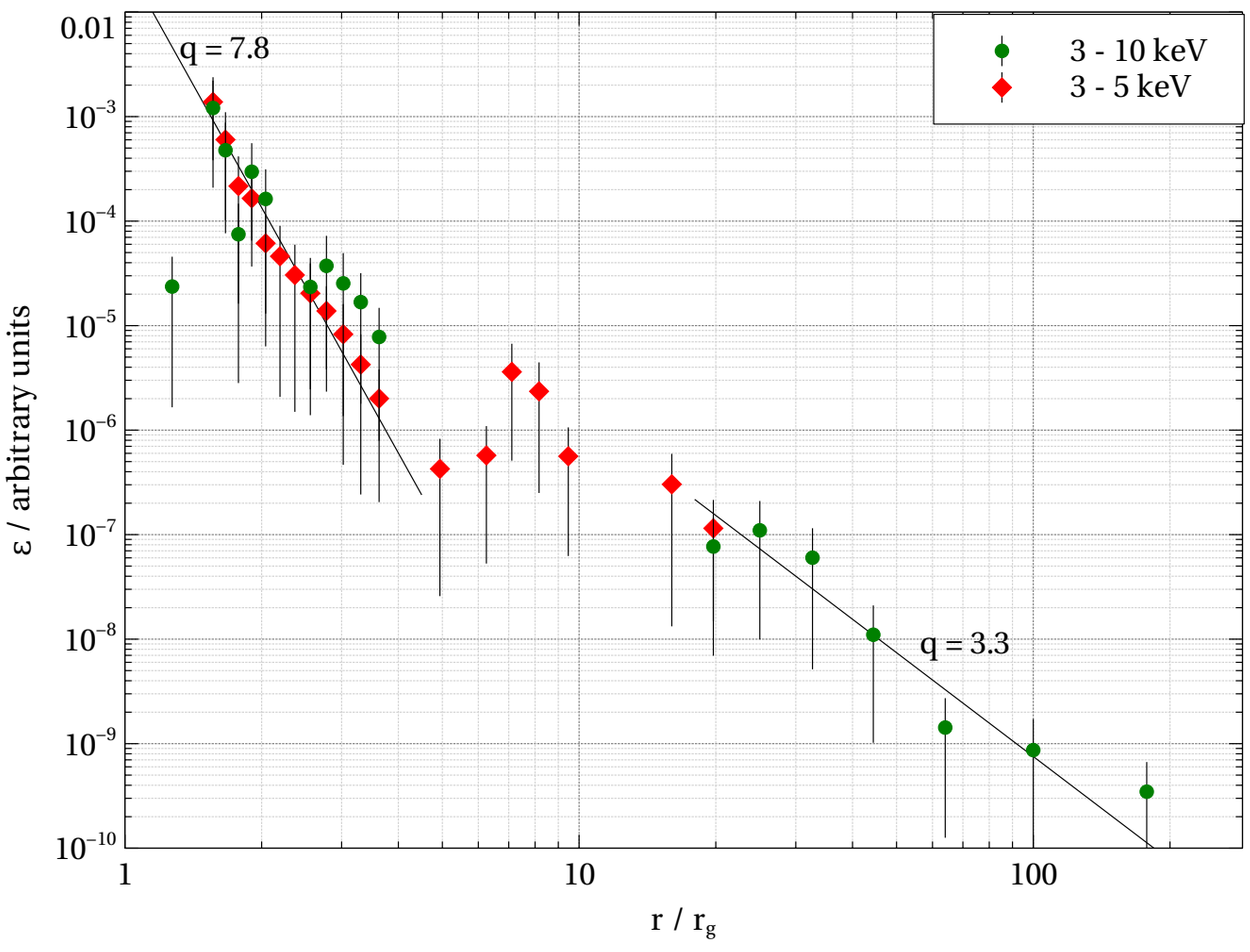
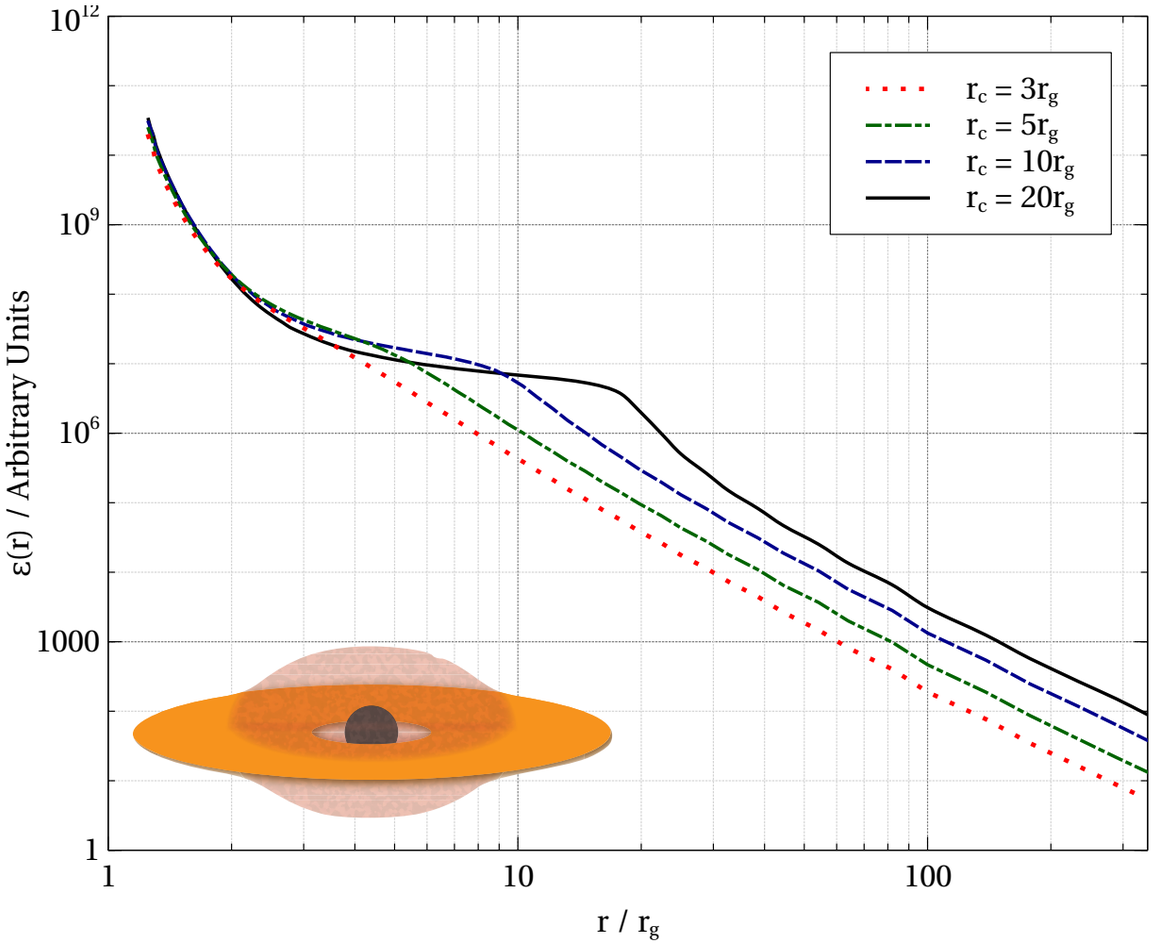
Finding the Corona





Wilkins & Fabian 2011, 2012, Wilkins & Gallo 2015

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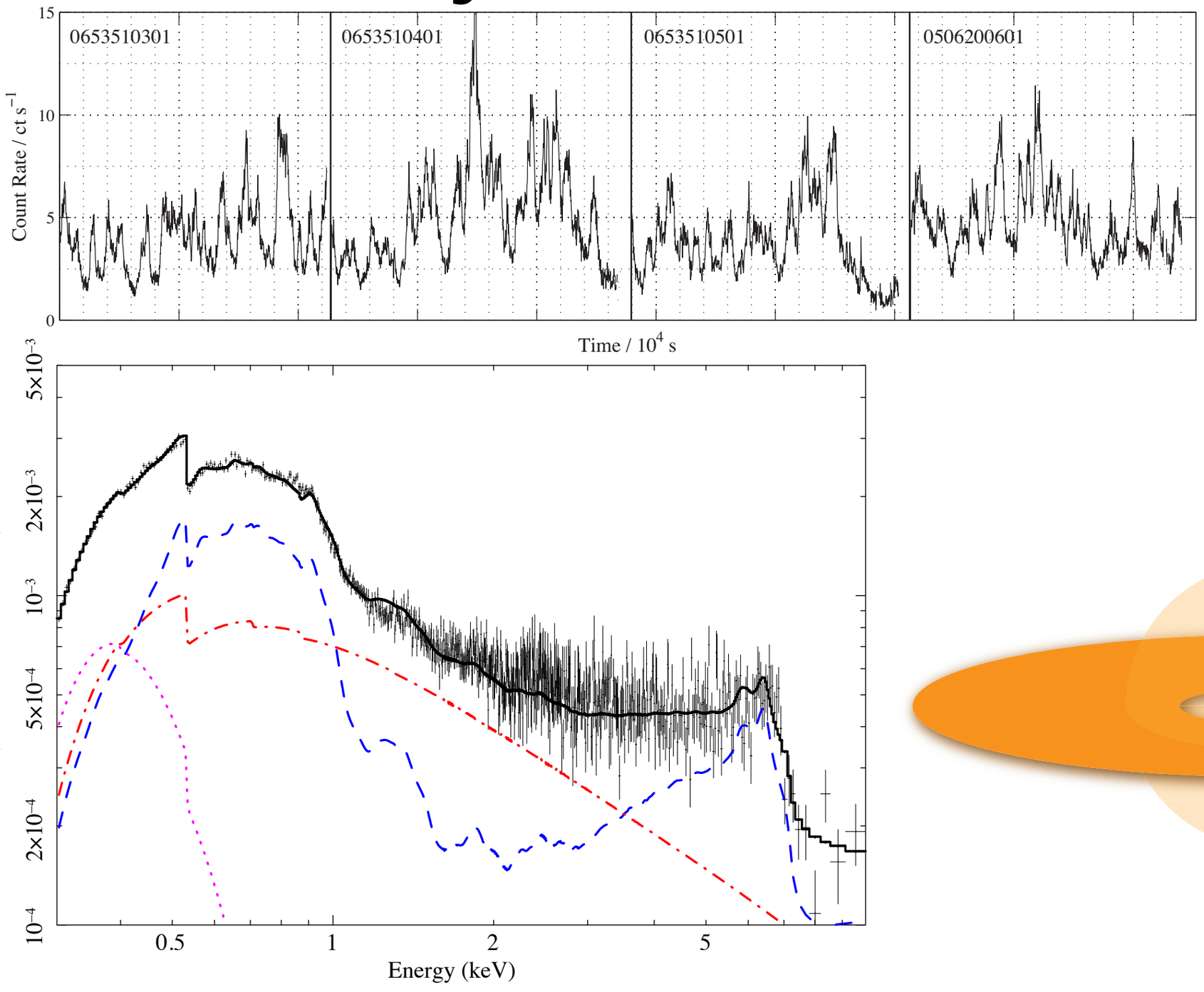
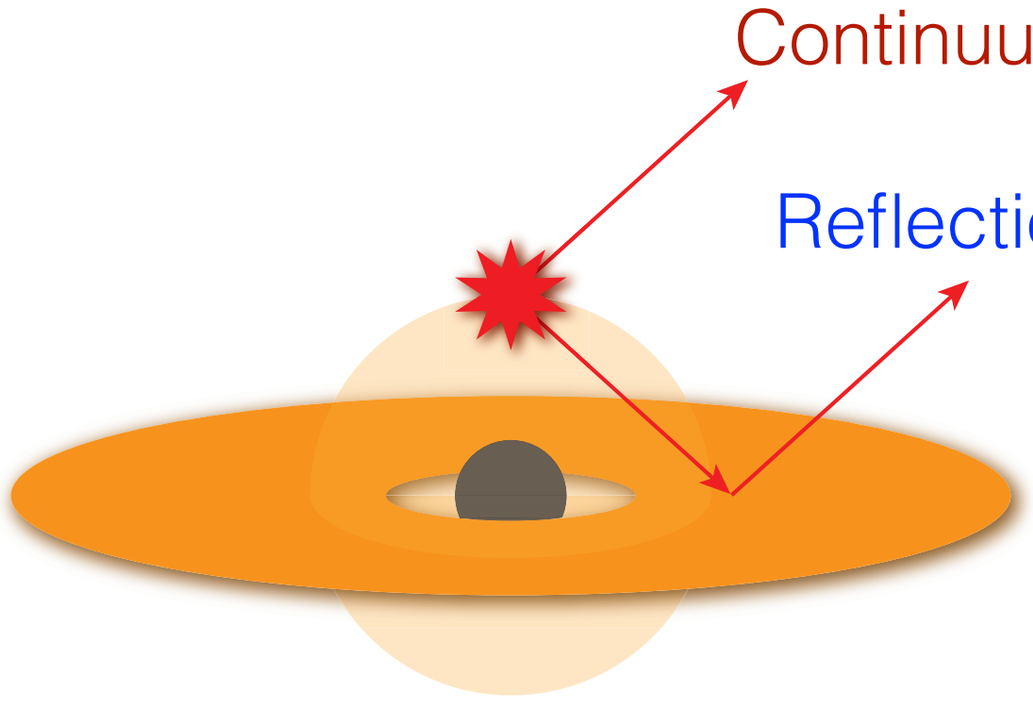
X-ray Reverberation





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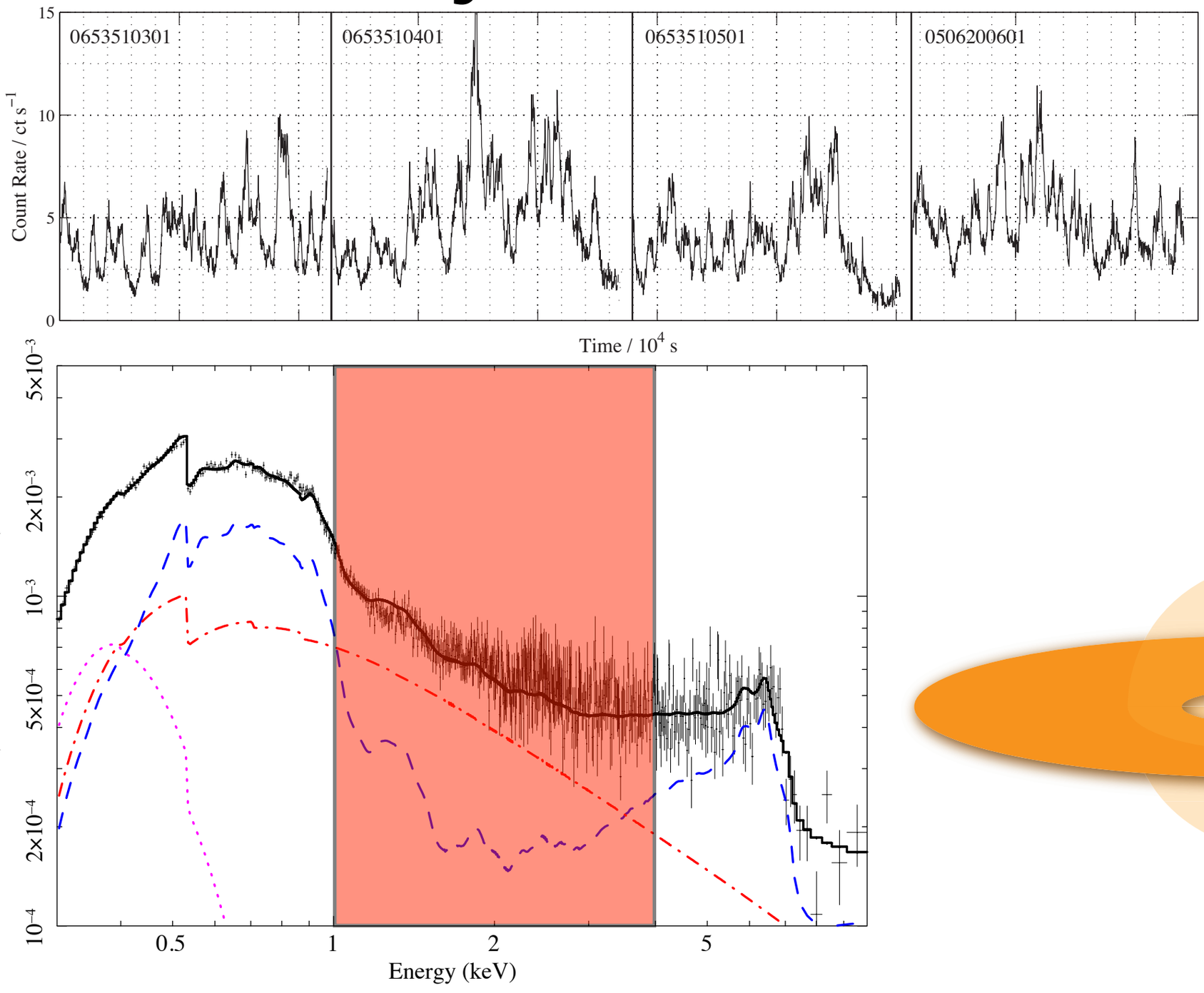

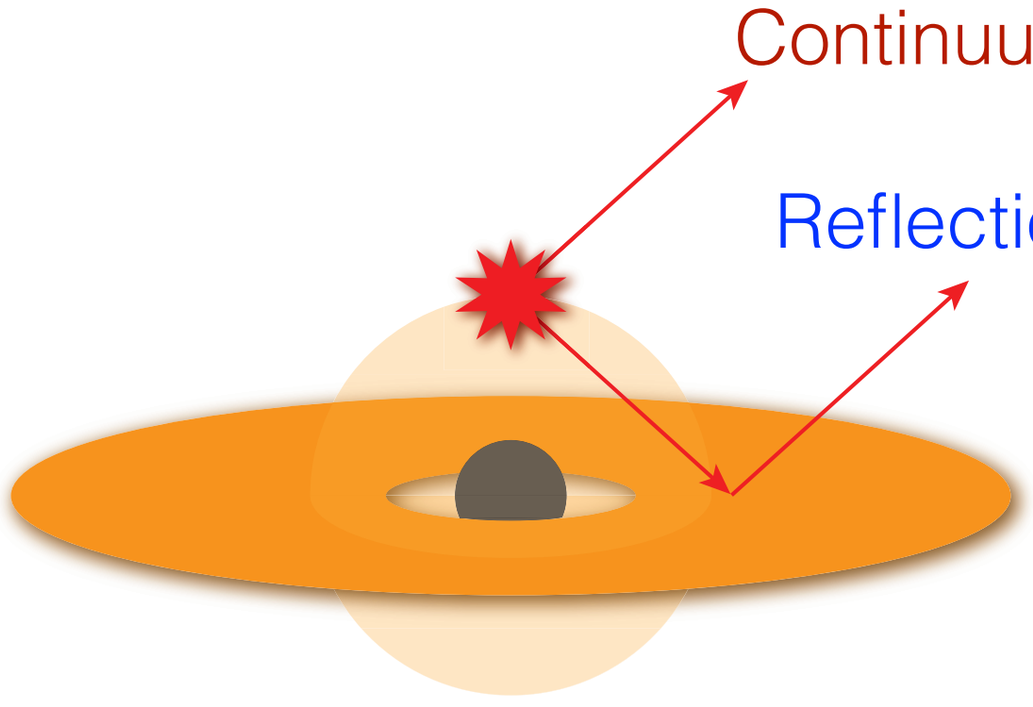

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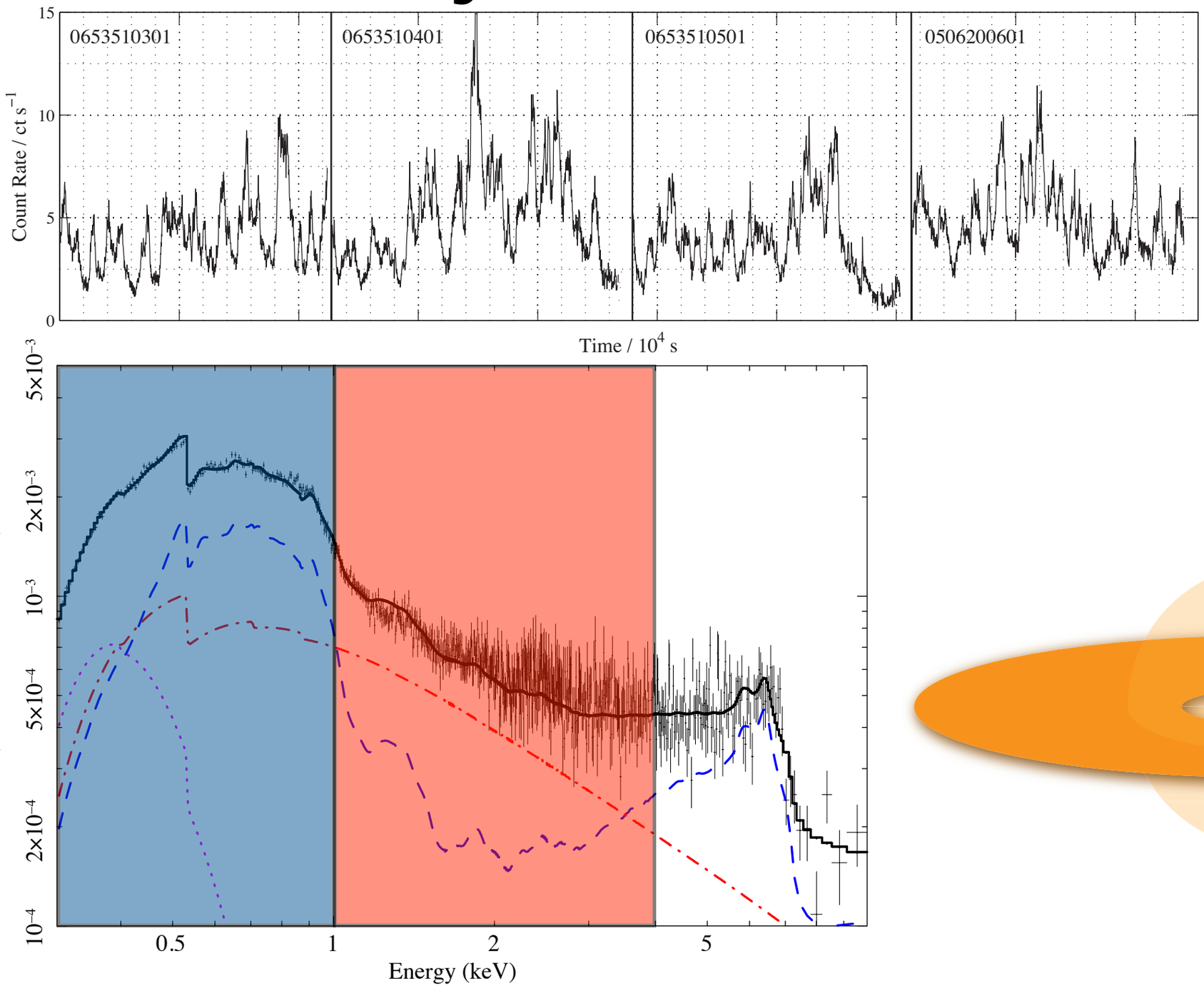

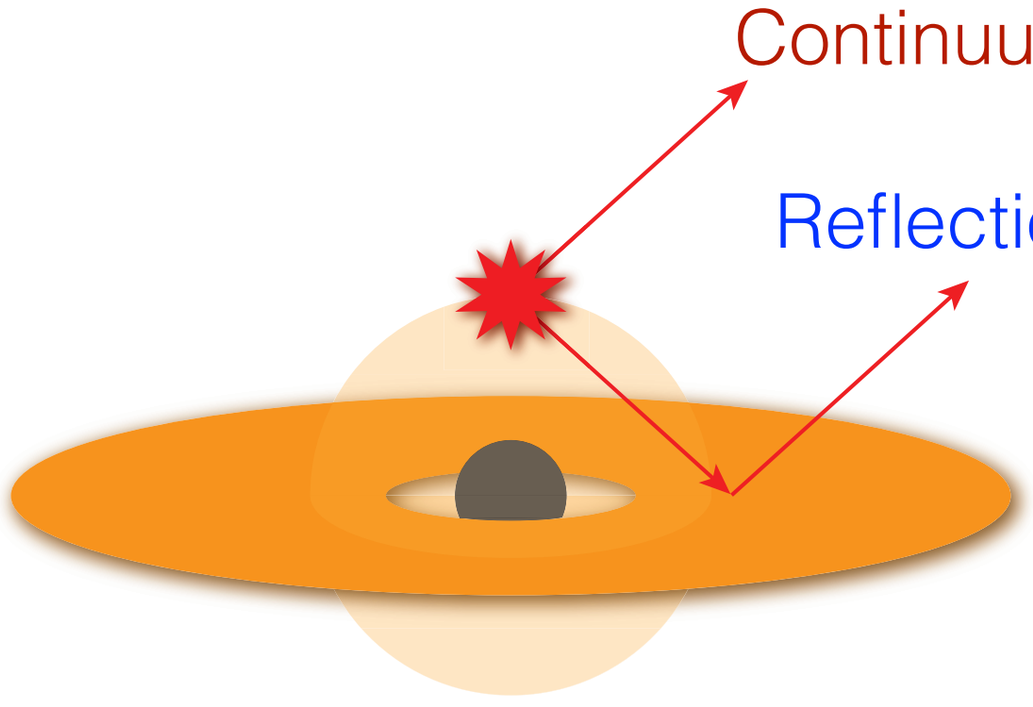

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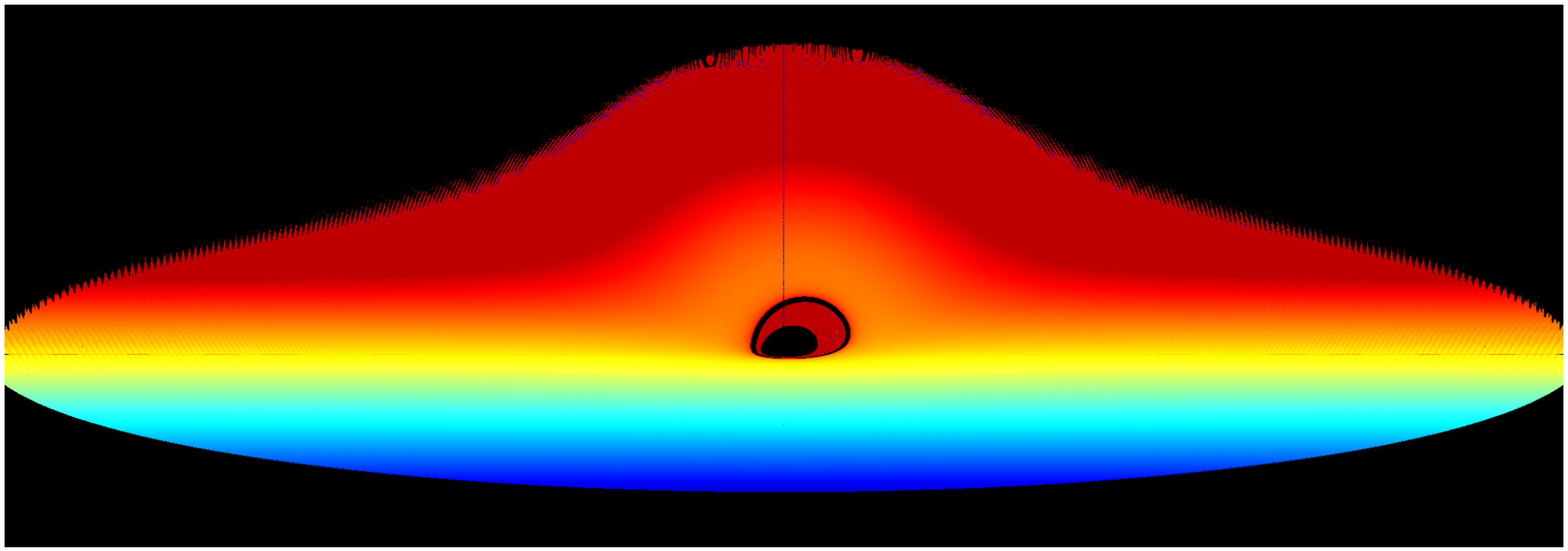
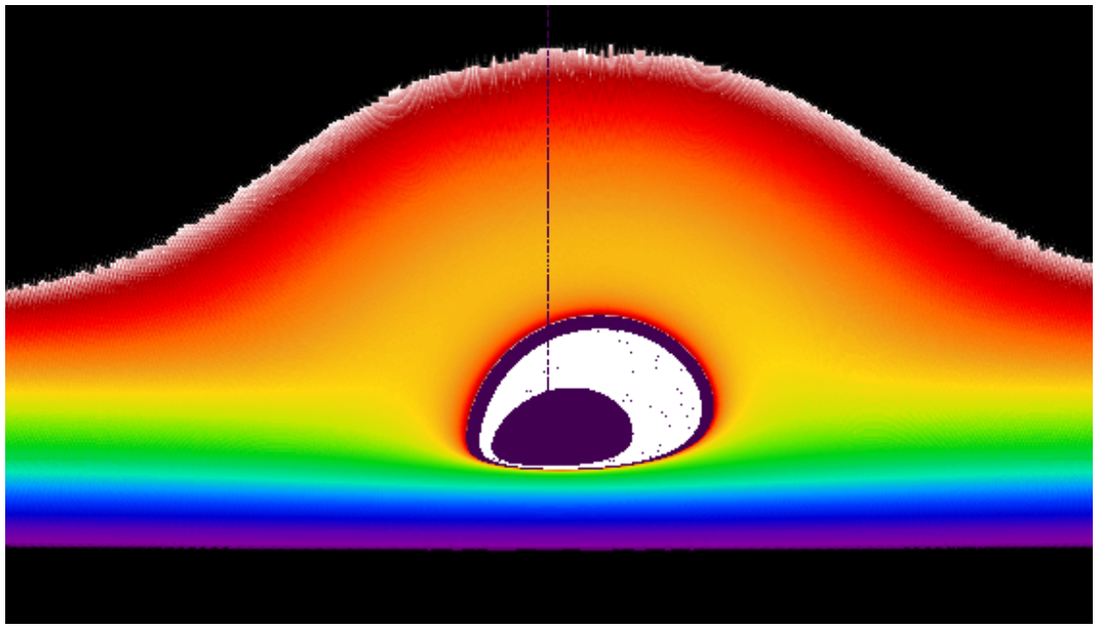
Ray Travel Time





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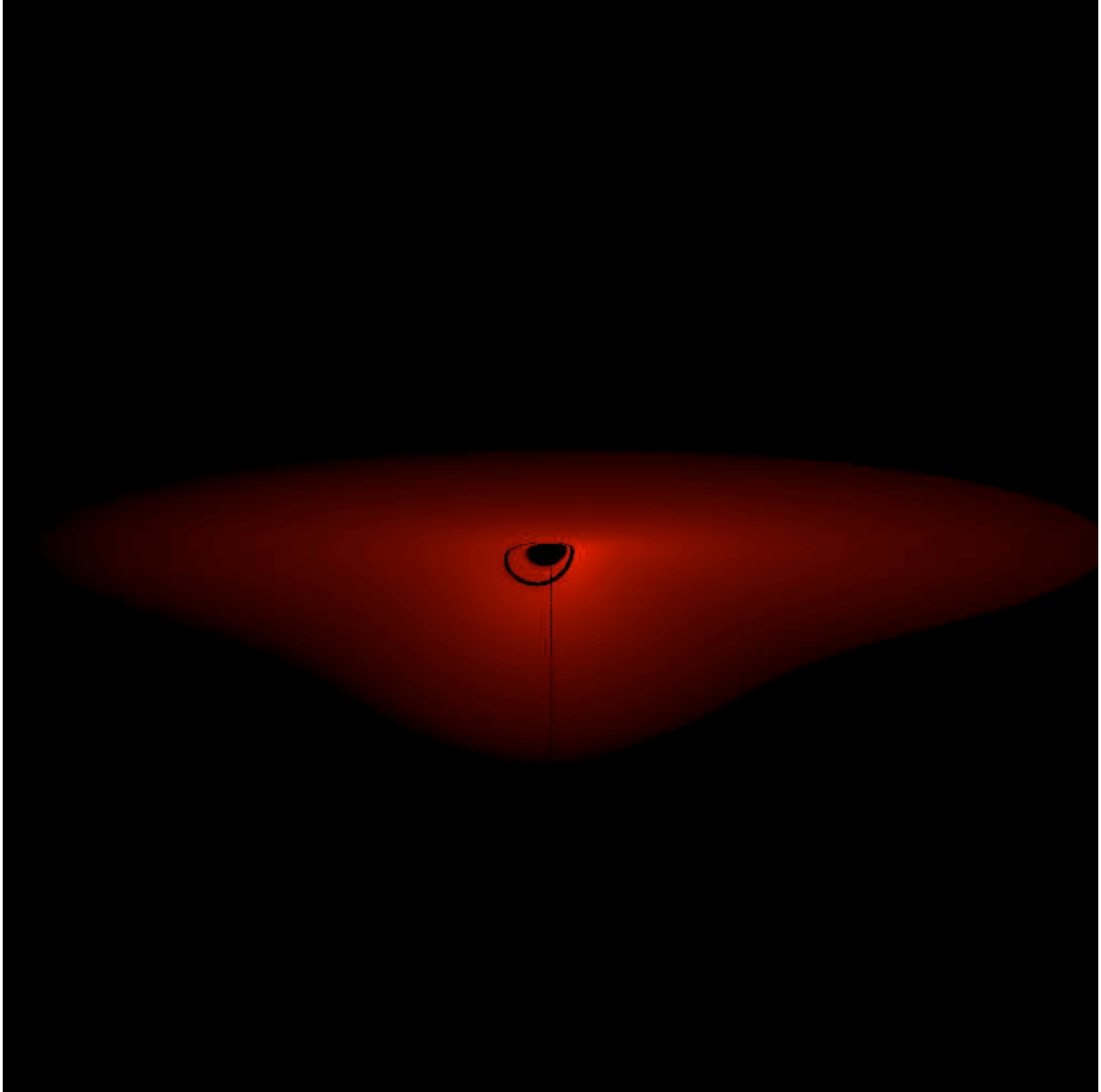

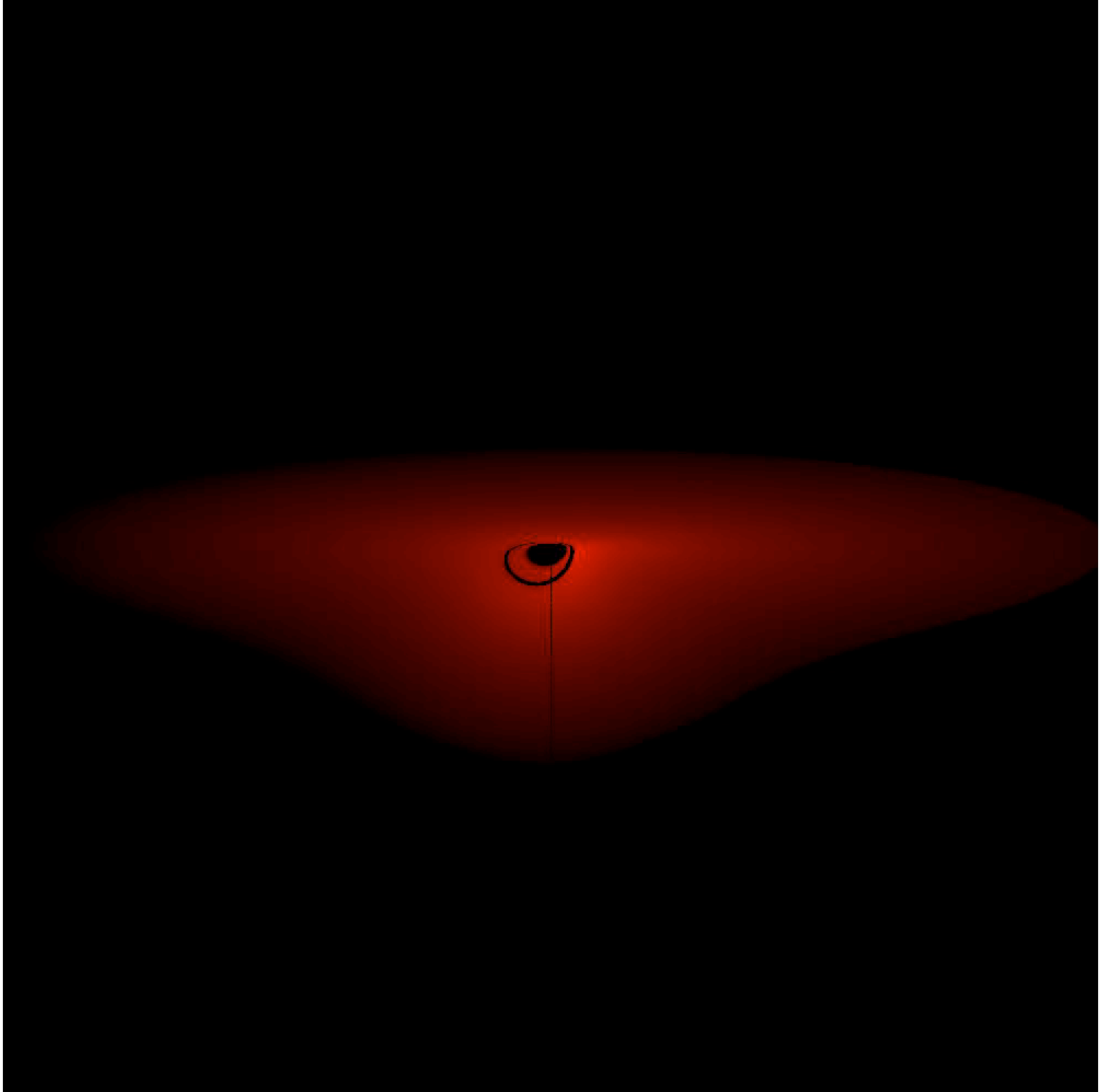

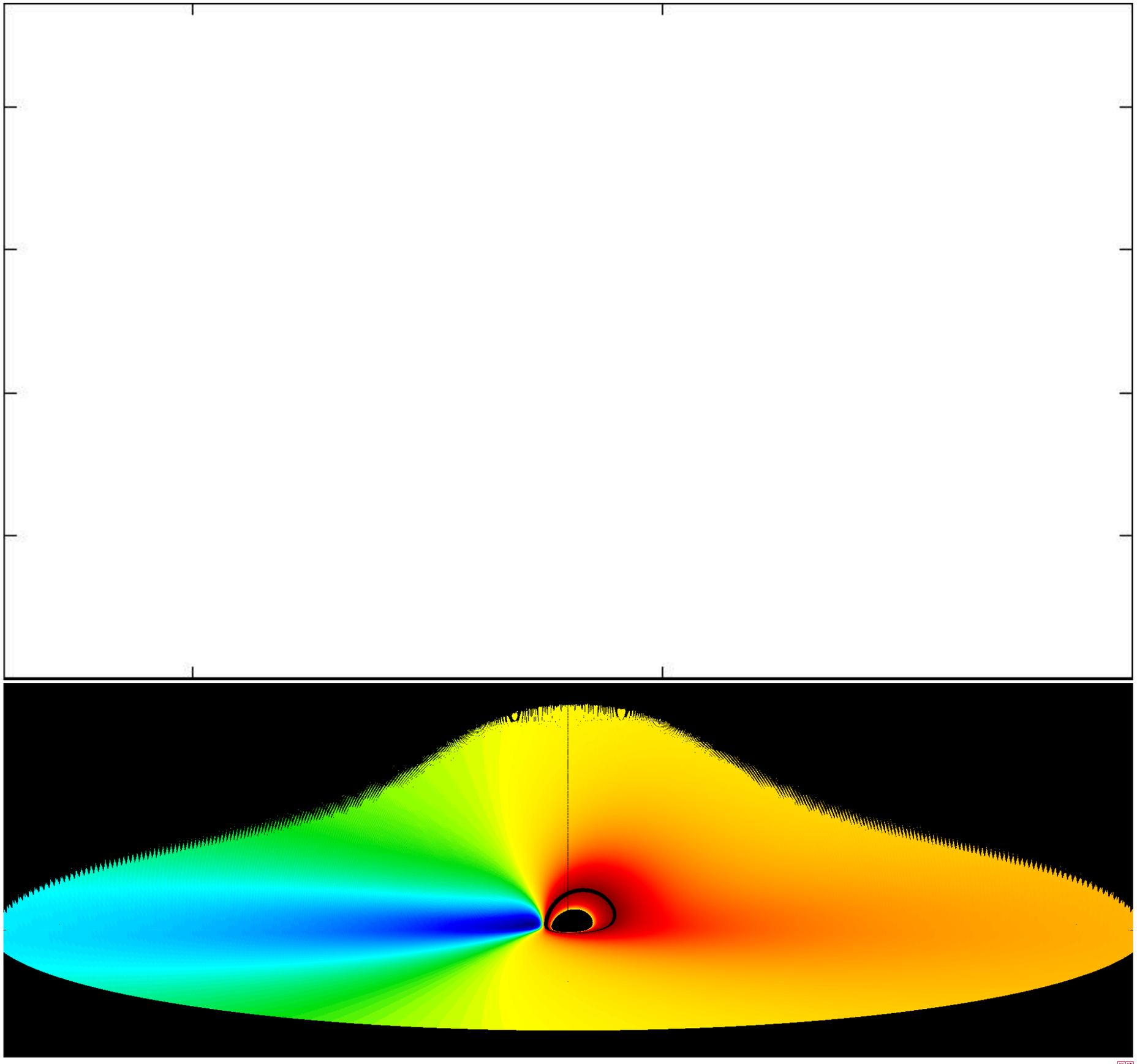

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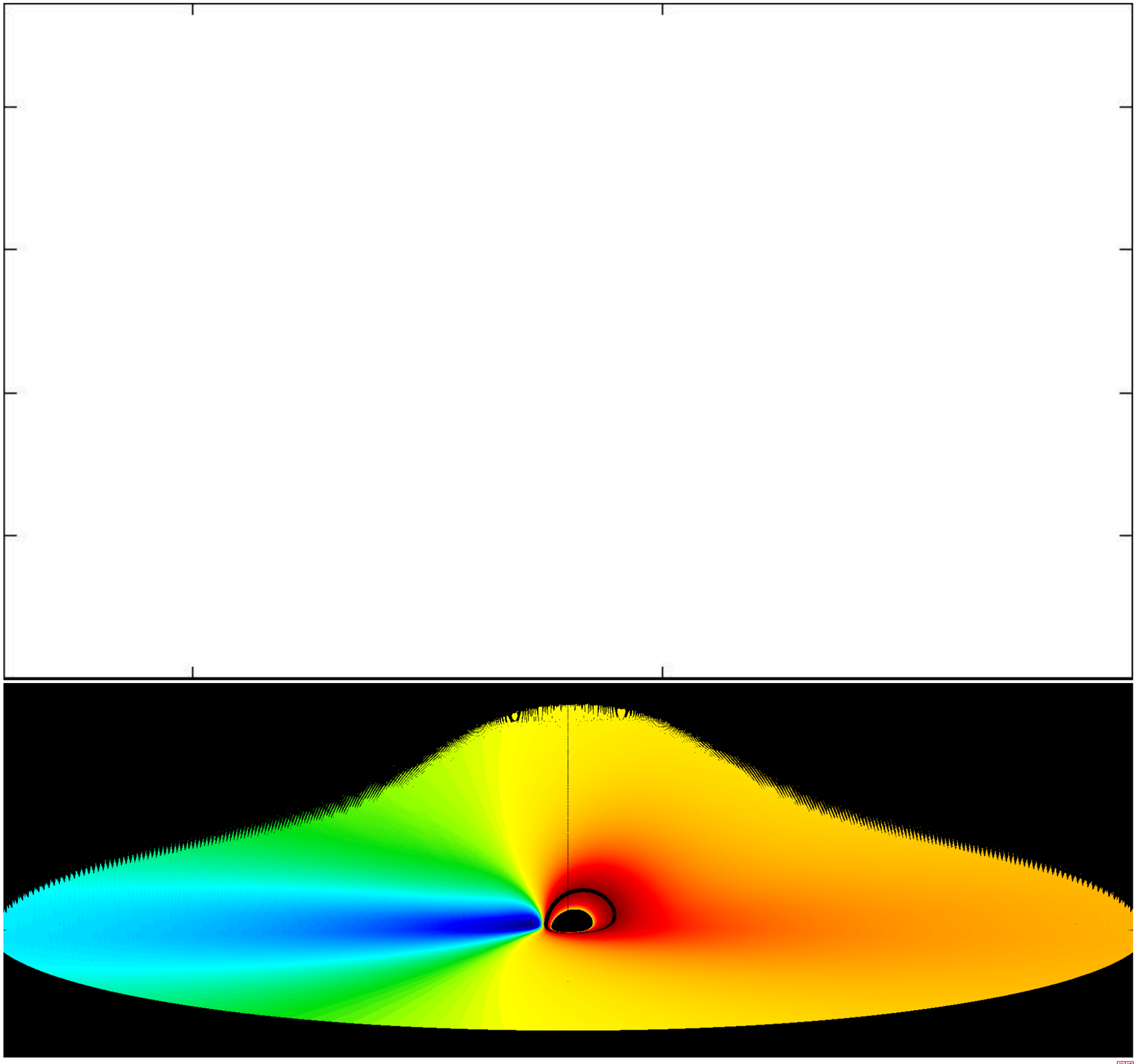

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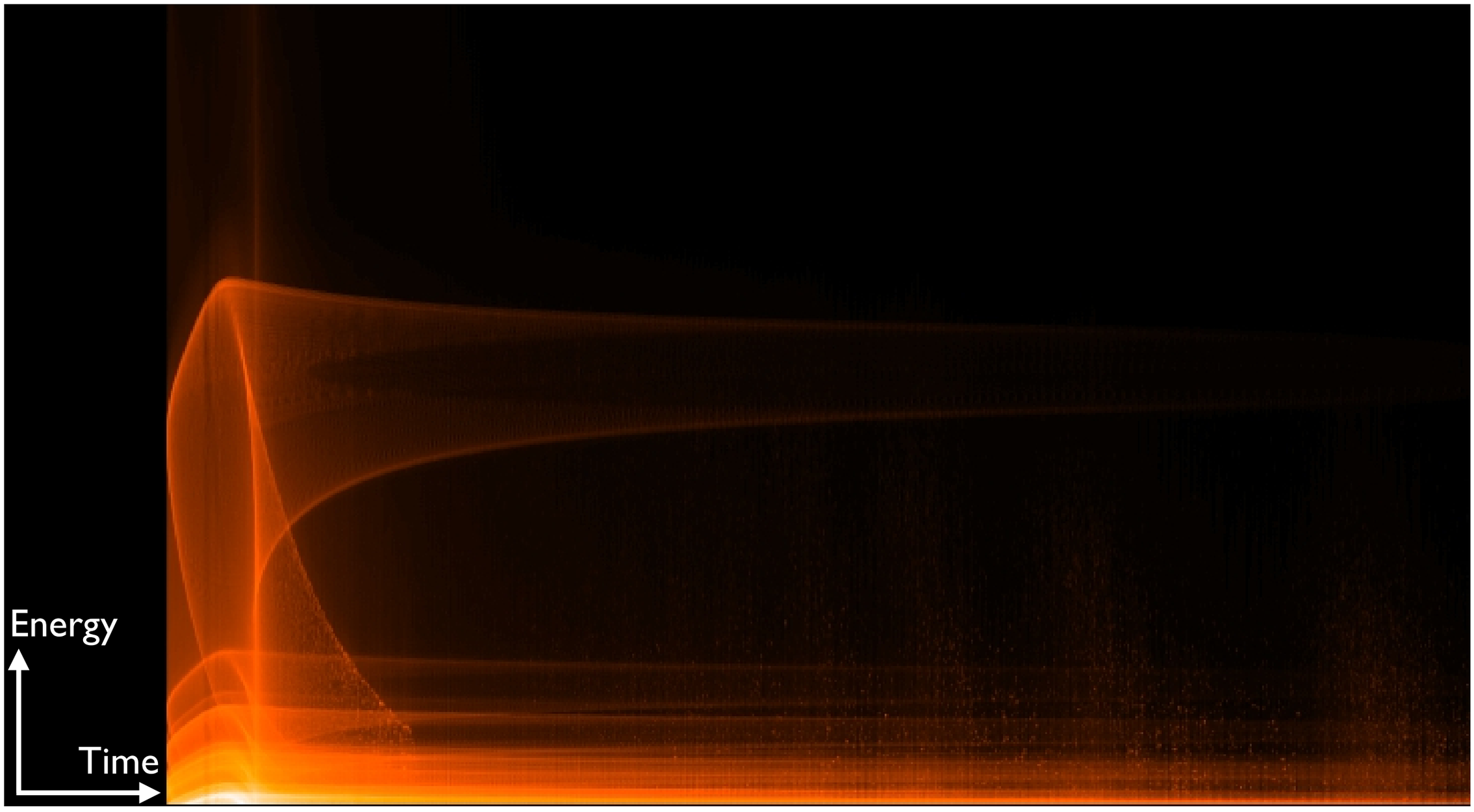
Reverberation Response



Reynolds et al. 1999, Wilkins et al. 2016, Uttley et al. 2014

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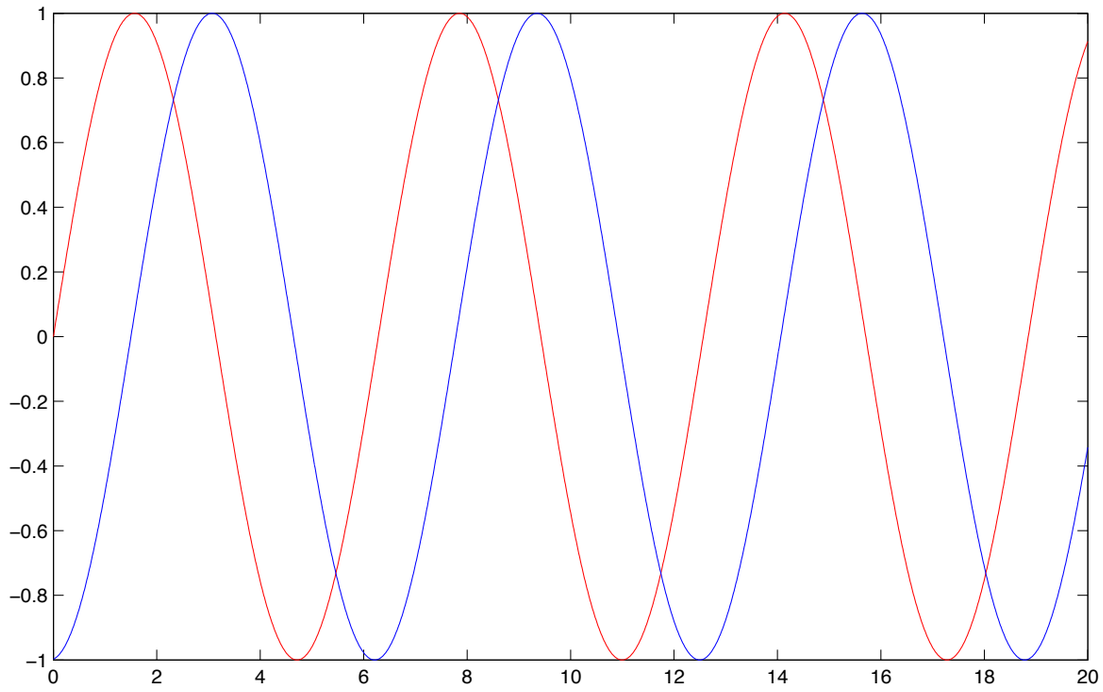
Measuring Reverberation



Take light curves in 2 bands (H dominated by continuum, S by reflection) and Fourier transform

$$\tilde{H}(\omega) = \left|\tilde{H}(\omega)\right| e^{i\varphi}$$

$$\tilde{S}(\omega) = \left|\tilde{S}(\omega)\right| e^{i\theta}$$

Fabian et al. 2009, Kara et al. 2013, Uttley et al. 2014

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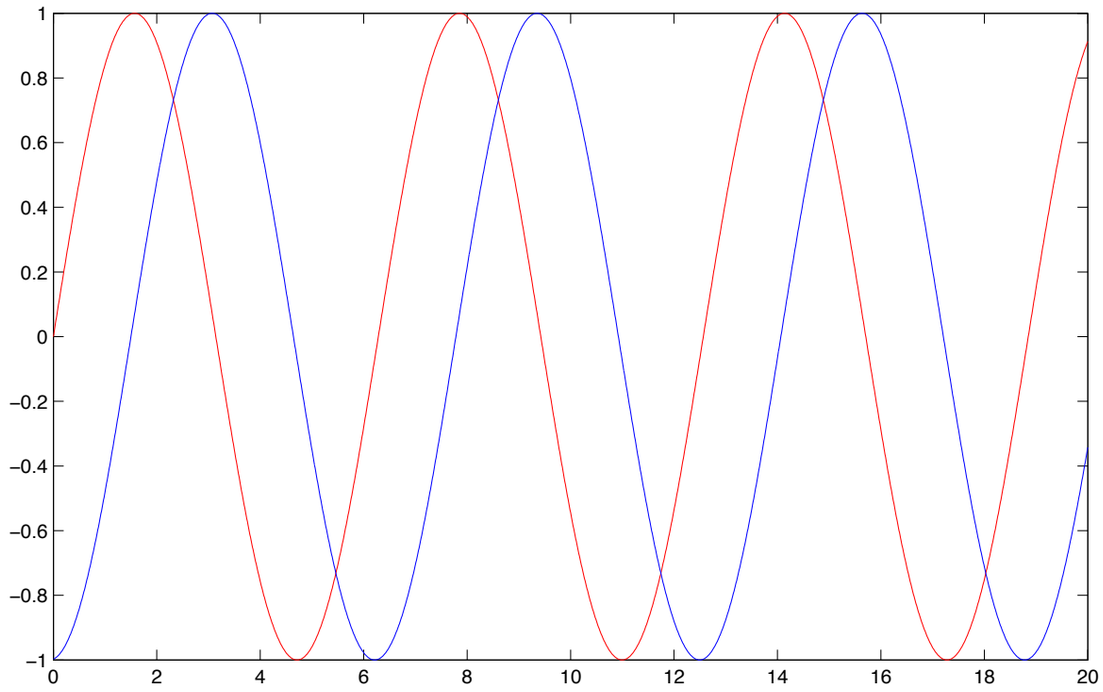

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The cross spectrum: $\tilde{C} = \tilde{S}^* \tilde{H} = \left|\tilde{S}\right| \left|\tilde{H}\right| e^{i(\varphi-\theta)}$

Fabian et al. 2009, Kara et al. 2013, Uttley et al. 2014

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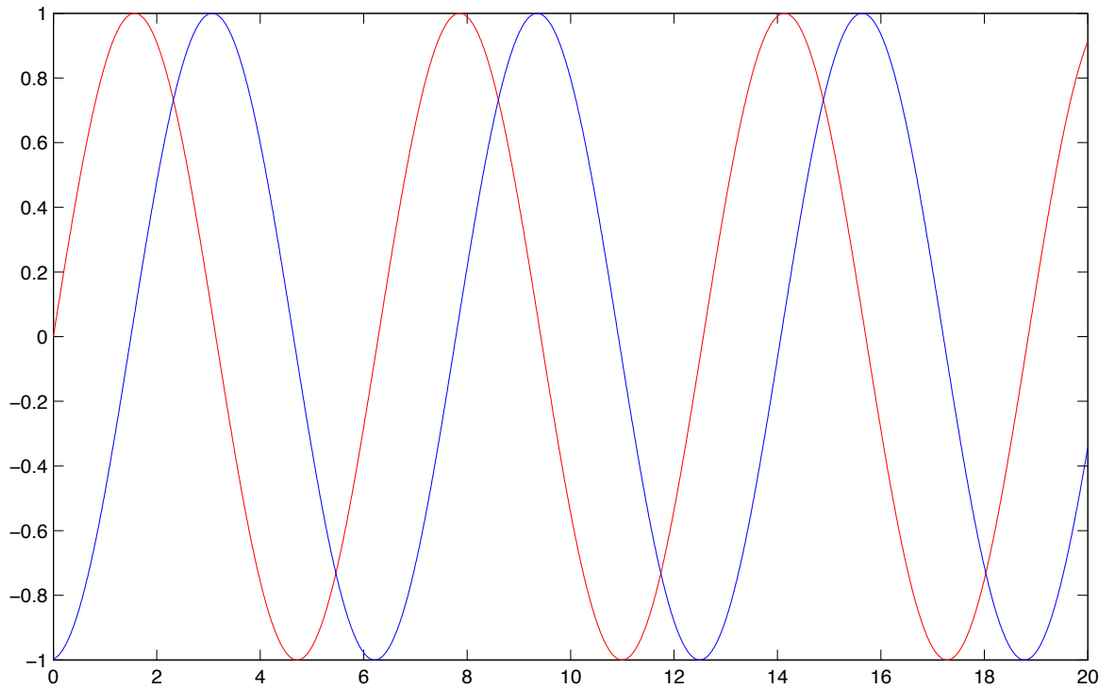

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And the time lag: $\varphi = \omega t$

Fabian et al. 2009, Kara et al. 2013, Uttley et al. 2014

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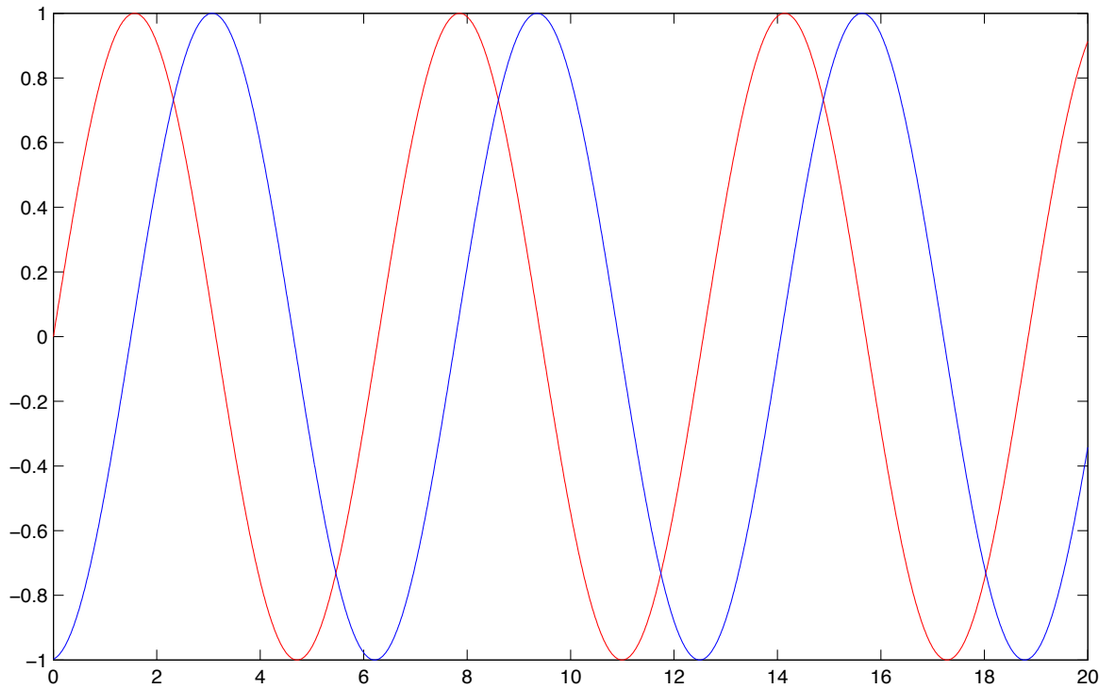
Measuring Reverberation



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And the time lag: $\varphi = \omega t \quad \therefore \tau(f) = \dfrac{\arg(\tilde{C}(f))}{2\pi f}$

Fabian et al. 2009, Kara et al. 2013, Uttley et al. 2014

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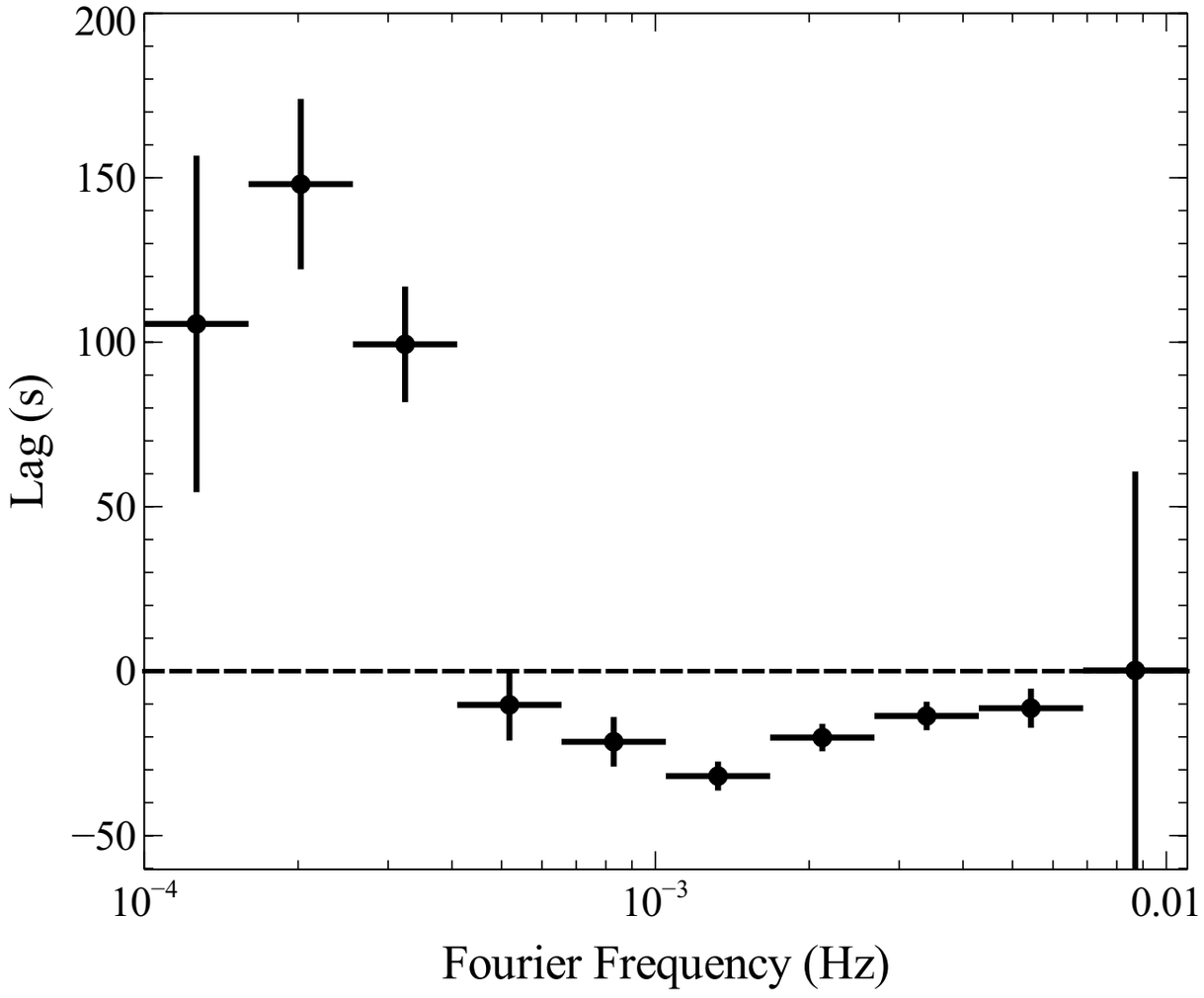

Fabian et al. 2009, Kara et al. 2013, Uttley et al. 2014

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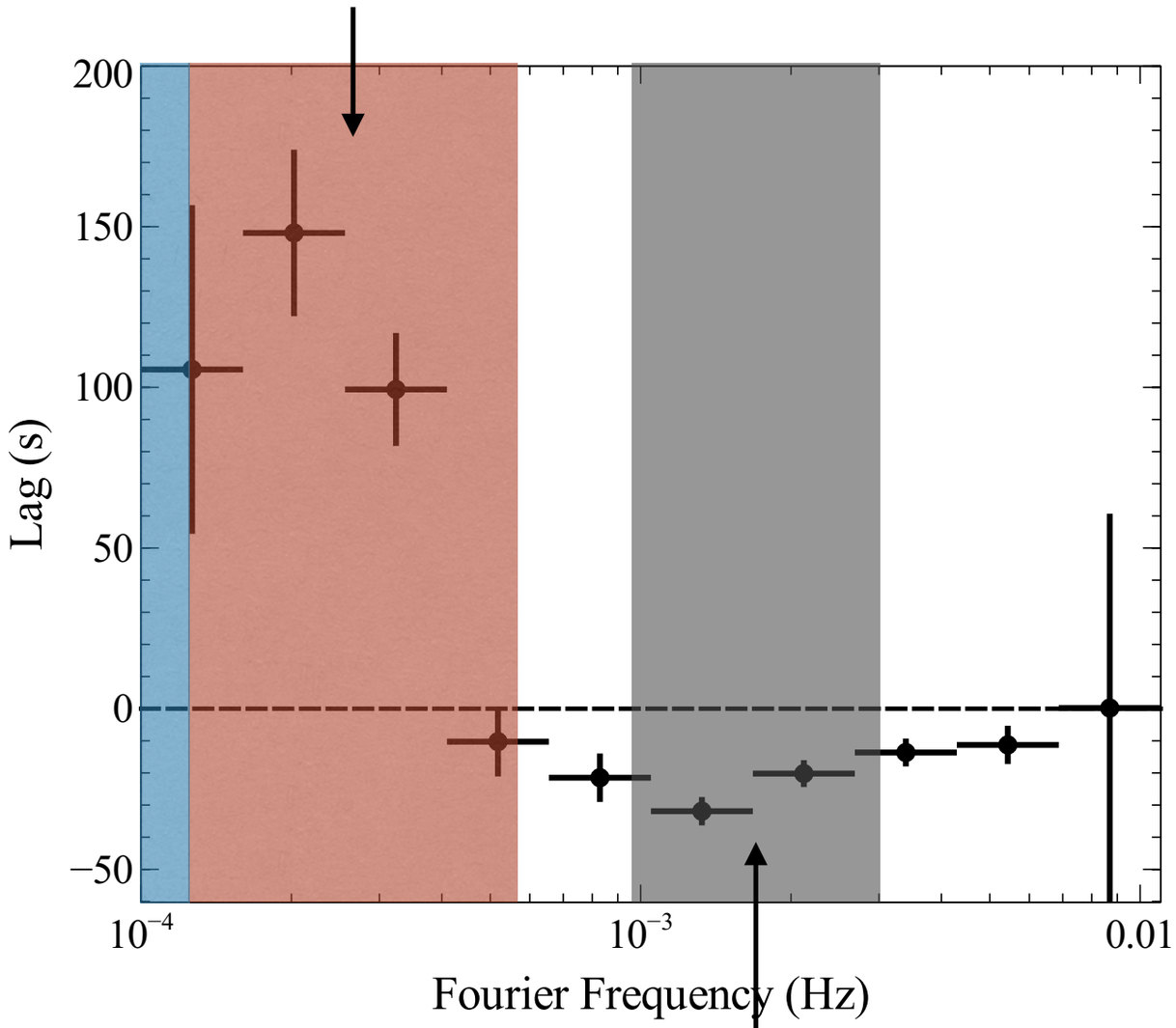
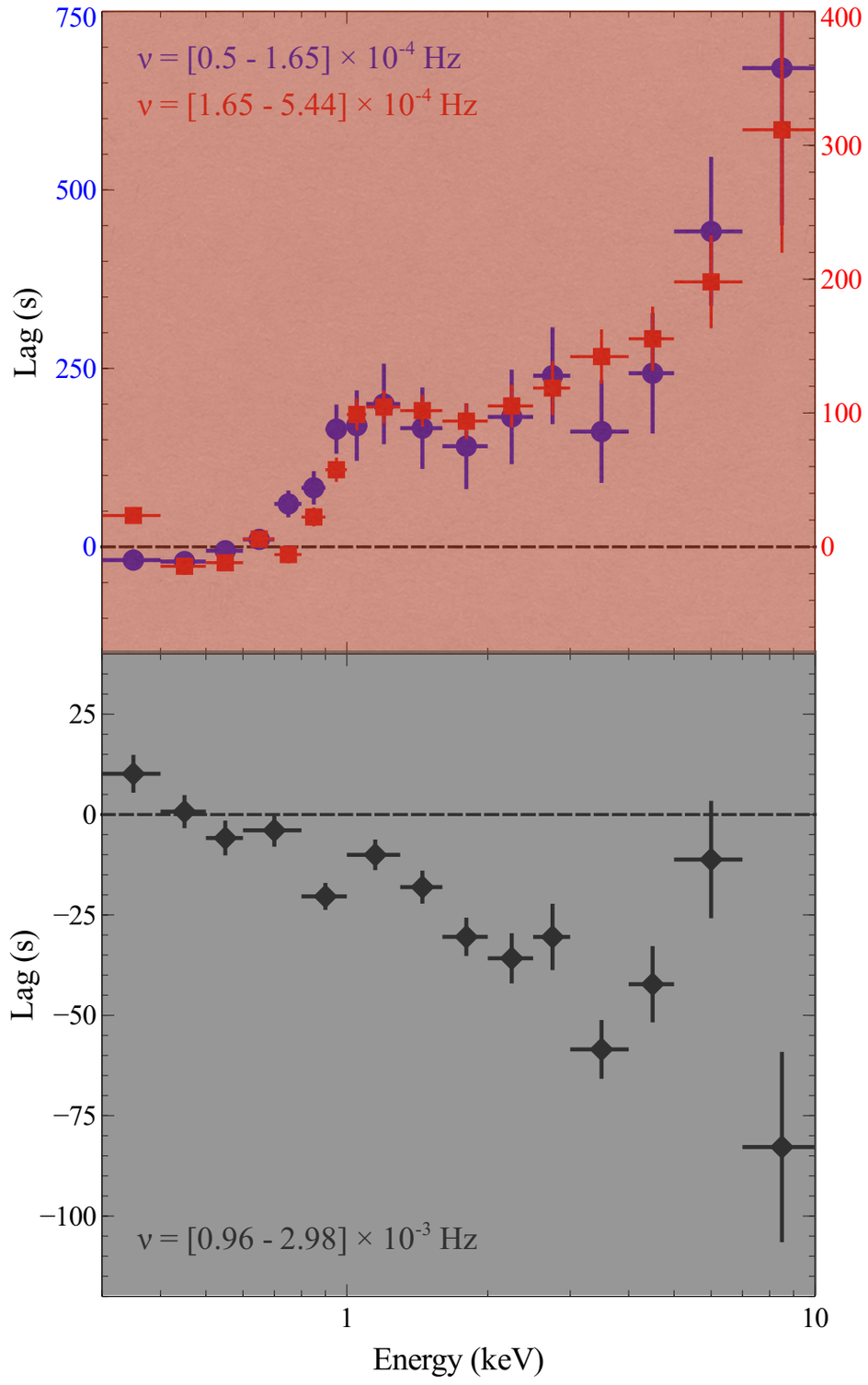
Hard X-rays lag soft (in continuum)



Soft (reflection-dominated) lags hard (continuum)



Fabian et al. 2009, Kara et al. 2013, Uttley et al. 2014

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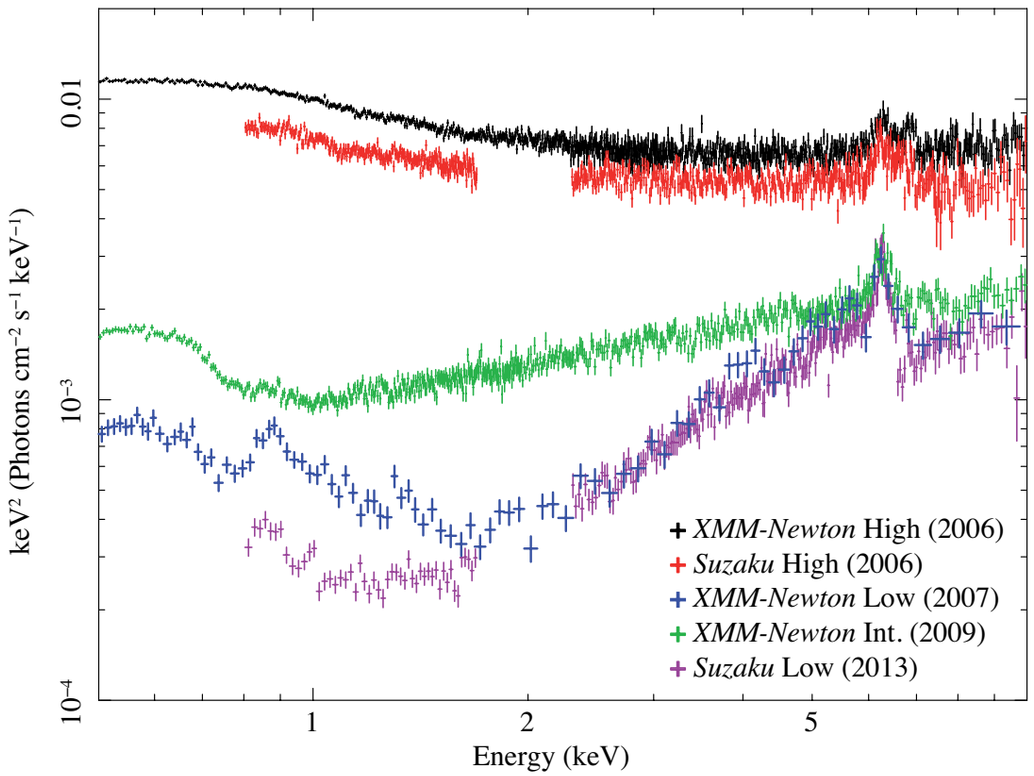
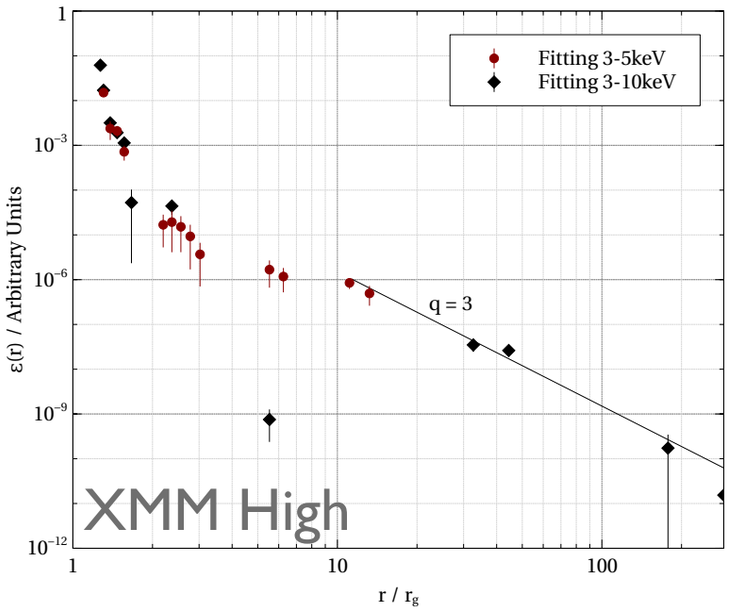
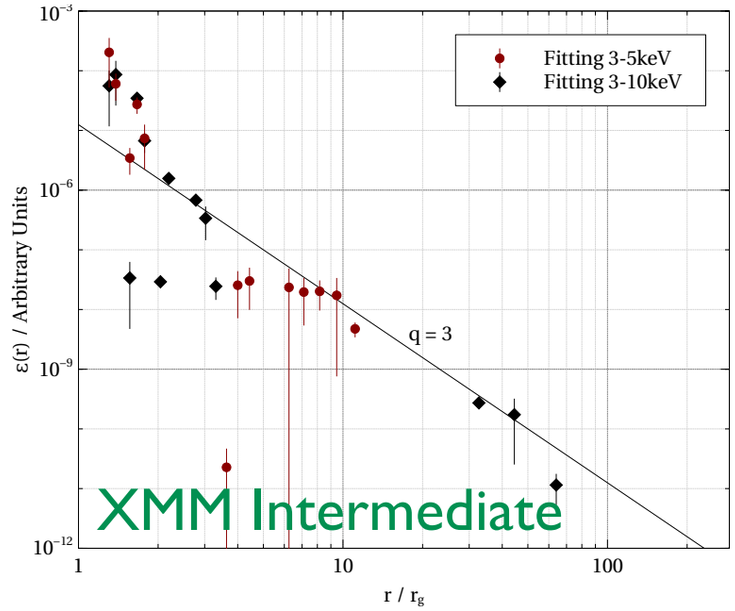
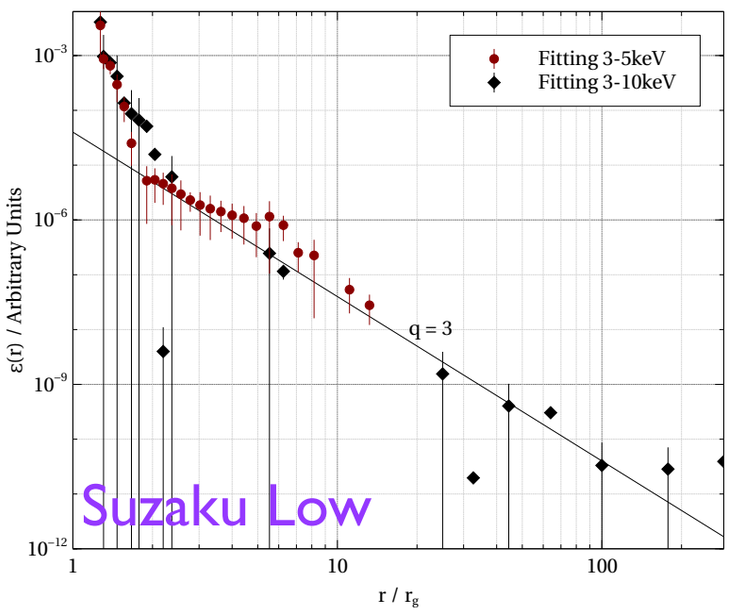
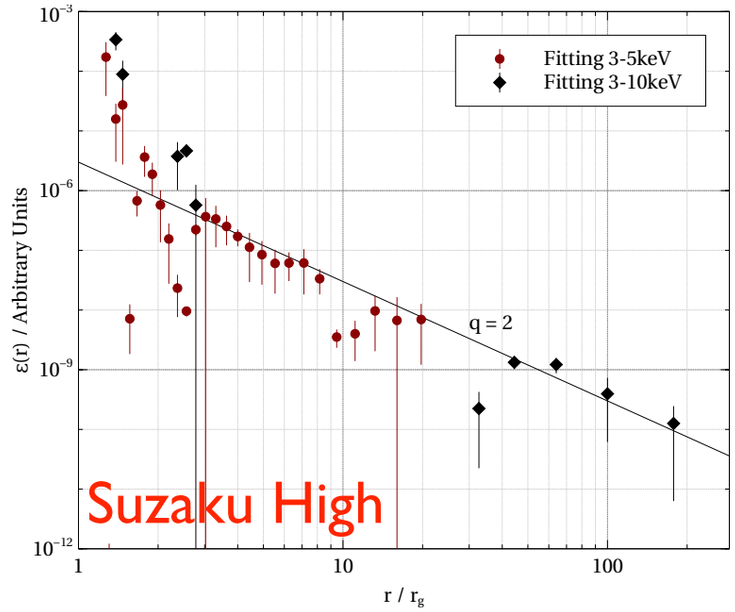
Variability in the Corona











Wilkins & Gallo 2015

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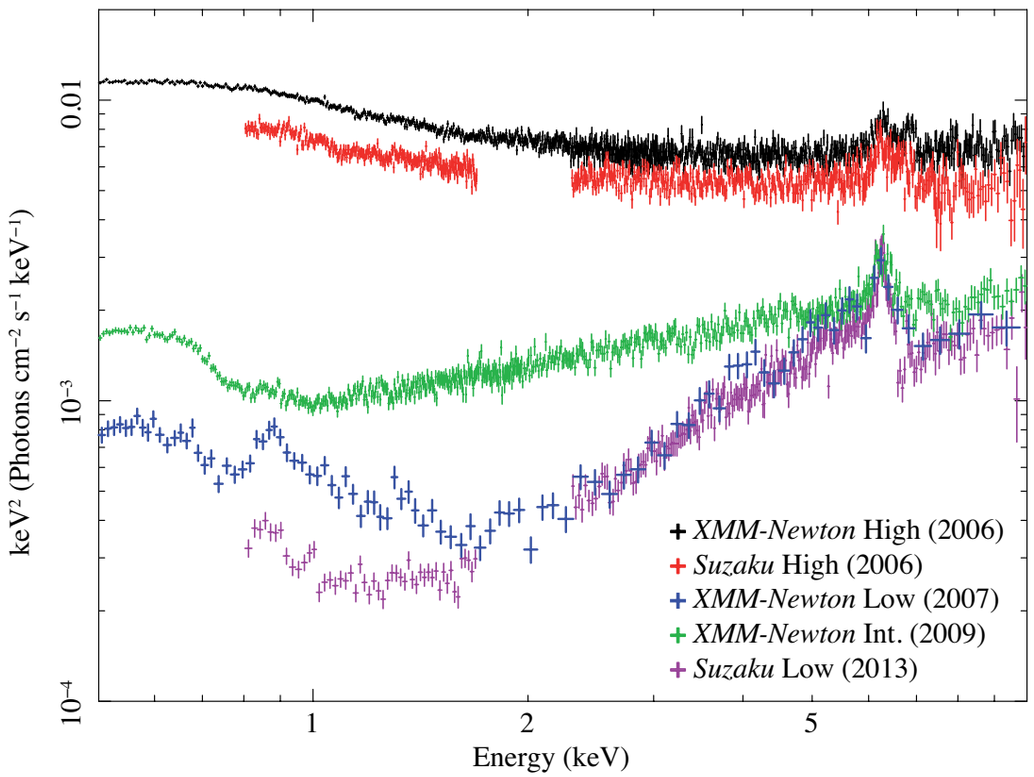
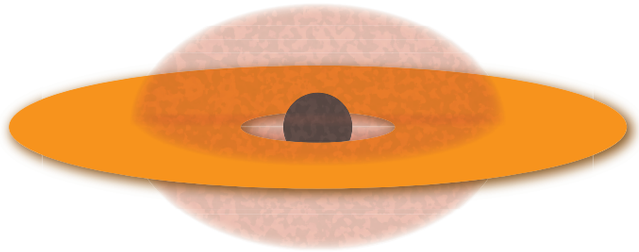
Variability in the Corona

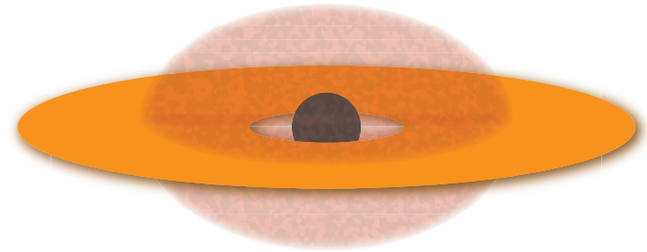


$r_b = 26^{+10}_{-7}\, r_g$



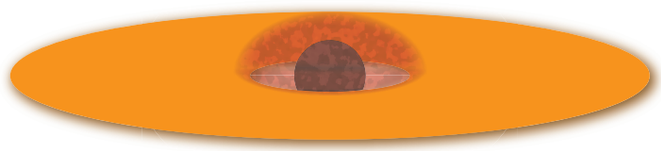
XMM High

$r_b < 12 r_g$

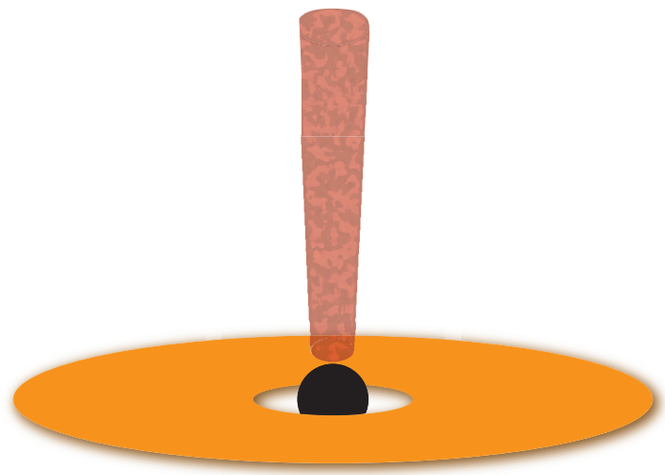


XMM Intermediate

$r_b < 5\, r_g$



Suzaku Low



Suzaku High

Wilkins & Gallo 2015

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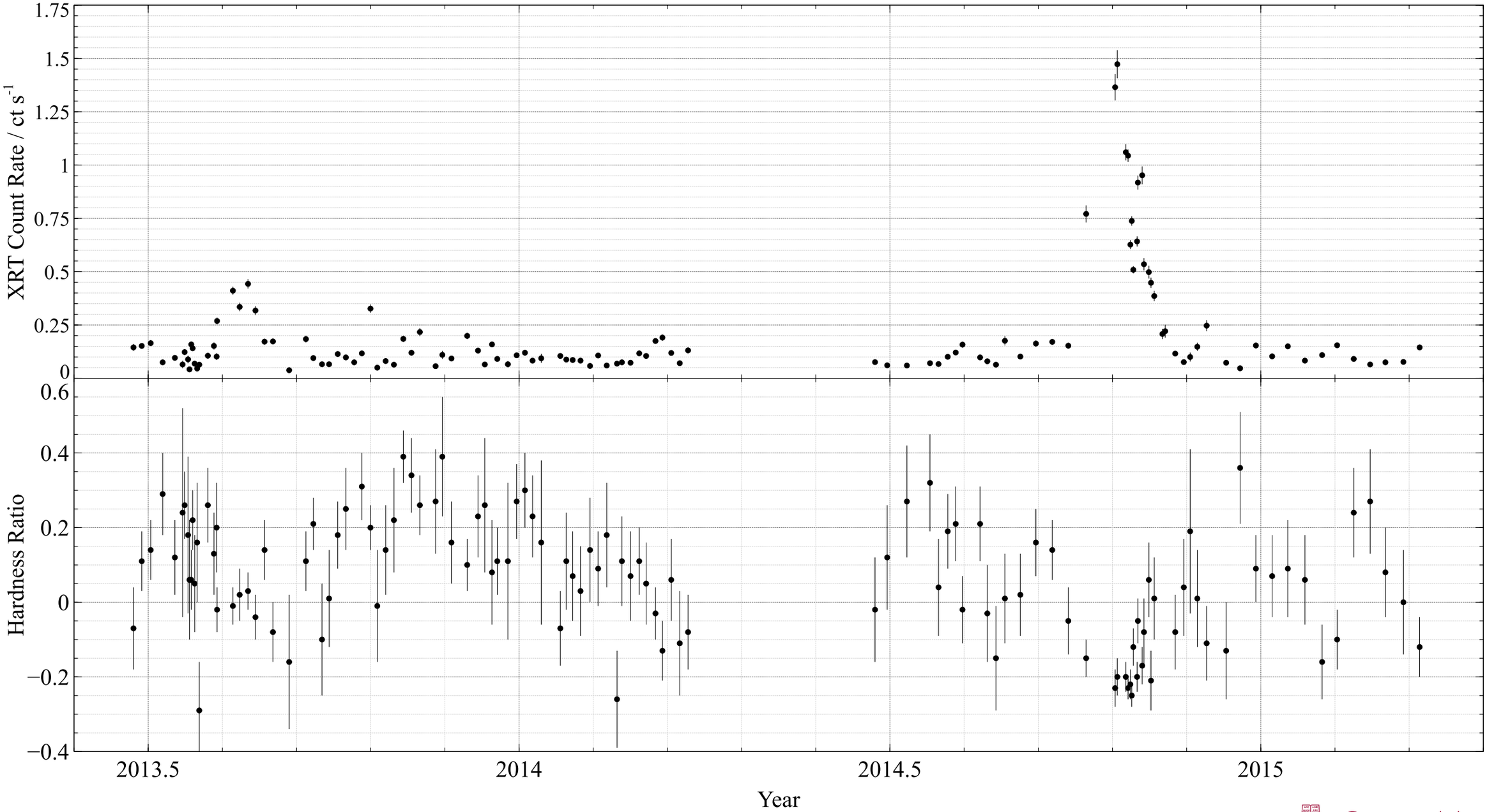
Flaring from the Corona



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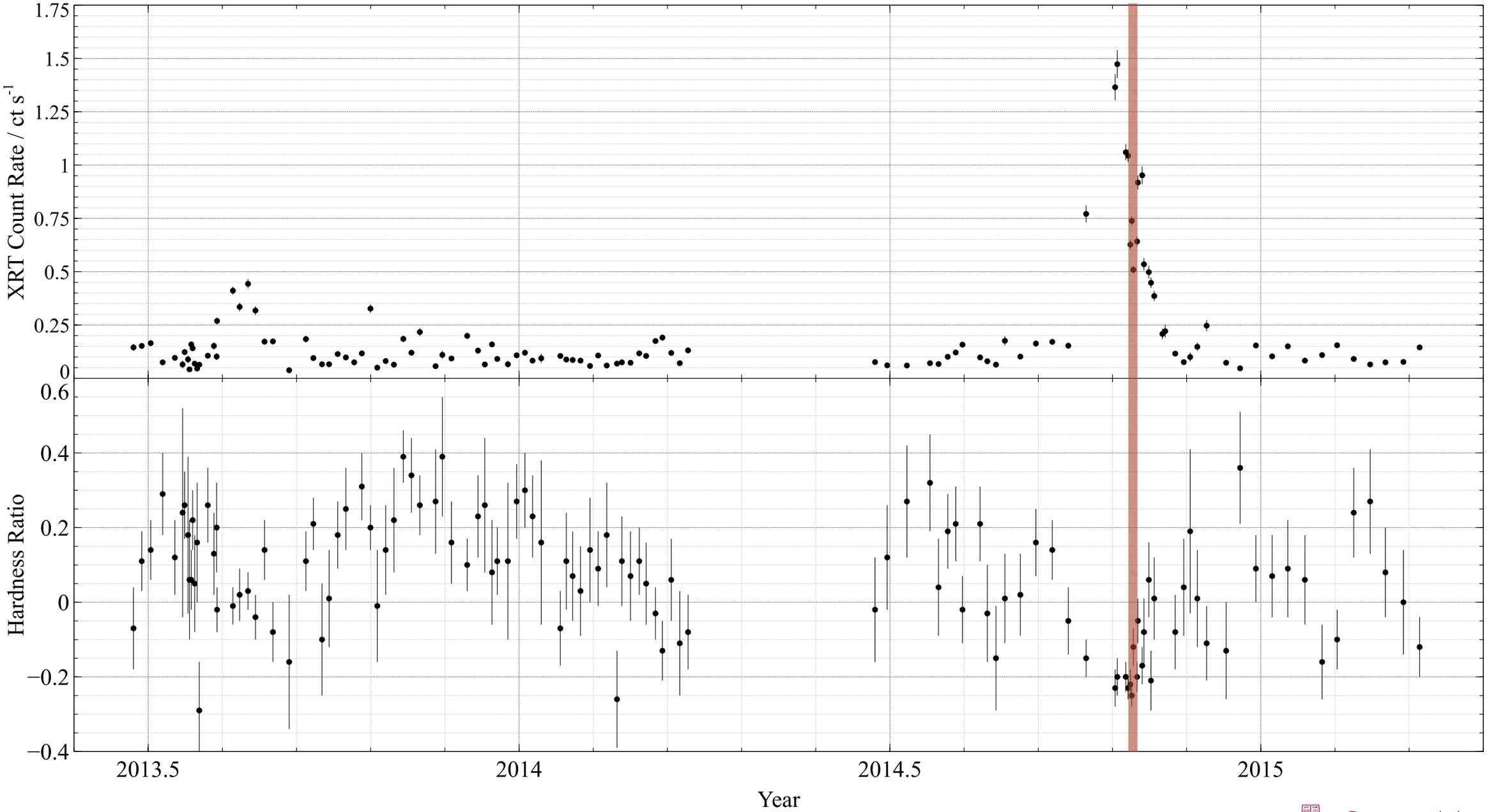
Flaring from the Corona



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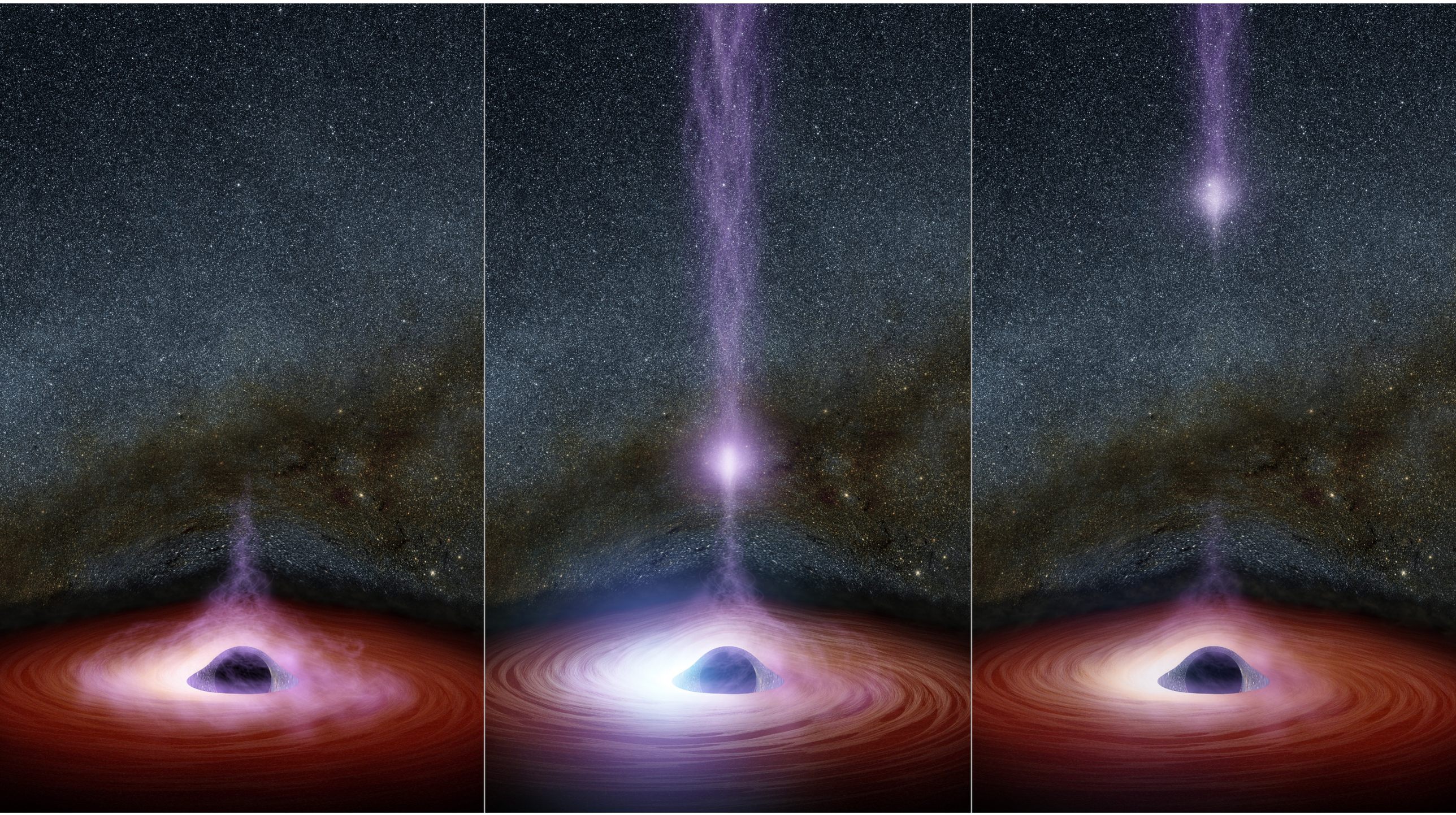

Wilkins et al. 2016

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Summary

- Space time around black holes described by general relativity
    - Particle orbits, passage of light, redshifts
- Accretion physics describes the inflow of gas and how luminous objects are powered
    - Simple models, MHD, GRMHD
- Theoretical models let us interpret observations
    - How are the most luminous objects in the Universe powered? How does the corona work? How does it vary? How is it governed over cosmic time?

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